Sixth Semester
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Manjusha

Core Course PHYSICS

Relativistic Mechanics and Astrophysics

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Preface

I feel extremely happy to present this book, **RELATIVISTIC MECHANICS AND ASTROPHYSICS** *for the benefit of the Core Course in Physics, Sixth semester B.Sc. Programme of University of Calicut. The book is prepared in accordance with the latest syllabus of the restructured, choice-based credit and semester system. Maximum care has been taken to maintain the standard and quality of the content, precision and perfection of the treatment, and the lucidity, simplicity and clarity of the style. All efforts have also been made for the sequential arrangement of the conceptual aspects of the subject matter in a systematic format, incorporating as much relevant facts as possible to make the description clear, straight-forward and self-explanatory. A large number of solved and unsolved problems, typical examples and all types of model questions, covering almost all fundamental aspects, have been given for an elaboration and systematic analysis of the basic concepts, and a deep insight into the subject matter.*

I do not claim the originality of the subject matter, since the essential materials have been collected from authentic reference sources. Despite our careful scrutiny, errors and omissions might have occurred and we will be highly obliged to those who bring them to our notice. All suggestions for improvement will be thankfully acknowledged and considered. I hope that the book will be highly useful to cater to the needs of the students.

Calicut, **P. Sethumadhavan**
November, 2021.

<u>UNIT ONE</u>

1 SPECIAL RELATIVITY

Introduction

The special theory of relativity, shortly called relativity, was put forward by Albert Einstein in the year 1905 which revolutionised the concept of space, time and motion on which Newtons laws were founded. Probably no physical theory in twentieth century has been the object of more discussion amongst philosophers and scientists, and at the same time caught the imagination of the intelligent layman, than the theory of relativity. This is essentially due to the fact that the concepts underlying the theory of relativity are not only radically new but also provide a frame work which embraces practically all the branches of the physical sciences. Actually everything stemmed from the inadequacy of Newtonian mechanics.

The entire edifice of mechanics was built upon the 3+1 laws of Newton. The first 3 stands for the three laws of motion and 1 stands for the law of gravitation of Newton. With these laws, Newton explained the macroscopic world with amazing success. It lead scientists to believe that these laws were universal in their applicability. To explain laws of motion, Newton assumed that space and time are absolute. But he could not support his conviction by any scientific argument, nevertheless he clung to this on theological grounds. But with the development of the wave theory of light scientists find it necessary to endow absolute space with certain mechanical properties. For this scientists evolved a hypothetical substance called 'ether' which they decided must pervade all space. It provided a mechanical model for all known phenomena of nature and a fixed frame of reference, the absolute space, which Newtons laws are required.

Michelson-Morley-Experiment

According to Christian Huygens wave theory, light propagates through ether which is stationary with respect to earth. So velocity of light should be different for different directions. To verify this Michelson and Morley used Michelson's interferometer. it consists of a semi silvered glass plate P, a compensating plate G, and two plane mirrors M_1 and M_2 kept perpendicular to each other. The compensating plate

G is kept to make the optical path travelled the same for both rays moving in perpendicular directions. Light from a monochromatic source 'S' on falling the plate P kept at angle of 45^0 gets split into two parts. One part undergoes reflection and goes towards M_1, at the same time the other part undergoes refraction through P and goes towards M_2. The rays get reflected from M_1 and M_2 enters an eyepiece T forming interference fringes. Let v be the velocity of the earth around the sun. Therefore with respect to the ether medium the apparatus has a velocity v. The effect is the same as if the apparatus is at rest and ether is moving in the opposite direction. As far as the reference frame lab is concerned the apparatus is at rest. So ether moves in the opposite direction with respect to the lab frame. Now consider a ray which has started towards M_1. Since the ether medium tracks this wave with it to reach M_1, light wave should be travelling in the direction PA so that the resultant velocity in ether is PM_1. Let c be the velocity of light when there is no relative motion between the source of light and ether. Therefore resultant velocity along $PM_1 = \sqrt{c^2 - v^2}$. Let L be the distance from P to M_1.

$\therefore$ The time taken by the light wave to travel from P to $M_1 = \dfrac{L}{\sqrt{c^2 - v^2}}$.

Time taken by the light wave to travel from M_1 to P will also be the same. Since in this case also the light wave is crossing the ether wind.

The time taken by the light wave to travel from P to M_1 and M_1 to P

$$t_1 = \frac{2L}{\sqrt{c^2 - v^2}} = \frac{2L/c}{\sqrt{1 - v^2/c^2}} \quad \text{...... (1)}$$

$v = 3\times10^4$ m/s (velocity of earth)

$c = 3\times10^8$ m/s

$$\therefore \quad \frac{v^2}{c^2} = \frac{9\times10^8}{9\times10^{16}} = 10^{-8}$$

Since $\frac{v^2}{c^2}$ is small, using Binomial approximation equation (1) can be written as

$$\left[\begin{array}{c}(1+x)^n \simeq 1 + nx \\ \text{when } x << 1\end{array}\right]$$

$$t_1 = \frac{2L}{c}\left(1-\frac{v^2}{c^2}\right)^{-1/2} = \frac{2L}{c}\left(1+\frac{v^2}{2c^2}\right) \qquad \text{...... (2)}$$

Velocity of the light wave which travels from P to M_2 is c-v (since it is in the opposite direction of the ether wind)

Velocity of light which travels from M_2 to P = c +v.

∴ Time taken by the light wave to travel from P to M_2 and M_2 to P.

$$t_2 = \frac{L}{c-v} + \frac{L}{c+v} = \frac{2Lc}{c^2 - v^2} = \frac{2L/c}{1-\frac{v^2}{c^2}}$$

(since $PM_2 = L$)

As before neglecting higher powers of $\frac{v^2}{c^2}$

$$t_2 = \frac{2L/c}{1-v^2/c^2} = 2L/c\left(1-\frac{v^2}{c^2}\right)^{-1} = \frac{2L}{c}\left(1+\frac{v^2}{c^2}\right) \qquad \text{....... (3)}$$

∴ The time difference between the two waves entering the telescope.

$$t_2 - t_1 = \frac{2L}{c}\left(1+\frac{v^2}{c^2}\right) - \frac{2L}{c}\left(1+\frac{v^2}{2c^2}\right)$$

$$t_2 - t_1 = \Delta t = \frac{Lv^2}{c^3} \qquad \text{....... (4)}$$

Equation 4 shows that there is a time difference between the two rays entering the telescope. A difference of time Δt causes a path difference

$$\Delta t \cdot c = \frac{Lv^2}{c^2} \qquad \text{....... (5)}$$

Because of this path difference interference fringes are formed in the telescope. Adjust the telescope and the cross wire is made to coincide with one of the fringes.

Now the apparatus is turned through 90° so that the two beams interchange their paths. i.e. in the rotated position the beam which was perpendicular to v now becomes parallel to v and viceversa, the path difference will be $= -\frac{Lv^2}{c^2}$.

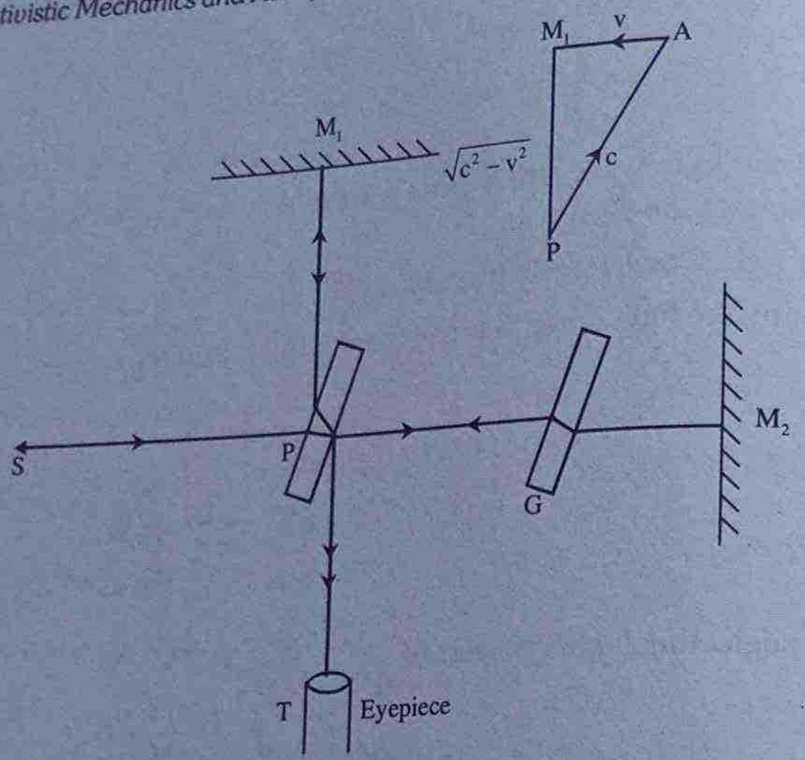


Figure 1.1

Displacement of the interference fringes

$$= \frac{Lv^2}{c^2} - -\frac{Lv^2}{c^2} = \frac{2Lv^2}{c^2}. \qquad \text{...... (6)}$$

But in the experiment no such fringe shift was observed. This shows that this experiment gives a negative result. This null result suggests that there is no ether medium which is stationary with respect to the earth and the velocity of light is a constant in all directions.

Example 1

In actual Michelson-Morley experiment the total distance from the partially silvered mirror to each of the mirrors was 10 m. The wave length of light used was 5000Å. If the orbital velocity of earth is taken as $30\,\text{km s}^{-1}$, calculate the total fringe shift when the apparatus is rotated through 90°.

Solution

$L = 10\,\text{m}$

$v = 30 \times 10^3\,\text{ms}^{-1}$

$\lambda = 5000 \times 10^{-10}$m

Fringe shift, $\delta = 2L\frac{v^2}{c^2} = \frac{2\times 10\times 9\times 10^8}{9\times 10^{16}} = 2\times 10^{-7}$m

Number of expected fringe shift $= \frac{\delta}{\lambda} = \frac{2\times 10^{-7}}{5\times 10^{-7}} = 0.4.$

Note : It may be noted that Michelson Morley experiment was sensitive enough to detect a fringe shift of the order of 0.01 fringe.

Example 2

If the arms of a Michelson interferometer have lengths l_1 and l_2. Show that the fringe shift when the interferometer is rotated by 90° with respect to the velocity v through the ether is $\delta = \frac{l_1 + l_2}{\lambda}\frac{v^2}{c^2}$, where λ is the wavelength of right.

Solution

See figure 1.1

The time taken by the light wave to travel from P to $M_1 = \frac{l_1}{\sqrt{c^2 - v^2}}$

Then time taken by the light wave to travel from M_1 to P will also be the same.

∴ The time taken by the light wave to travel from P to M_1 and M_1 to P

$$t_1 = \frac{2l_1}{\sqrt{c^2 - v^2}} = 2l_1(c^2 - v^2)^{-\frac{1}{2}}$$

$$t_1 = \frac{2l_1}{c}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \frac{2l_1}{c}\left(1 + \frac{v^2}{2c^2}\right)$$

Time taken by the light wave to travel from P to M_2 and M_2 to P

$$t_2 = \frac{l_2}{c - v} + \frac{l_2}{c + v} = \frac{2l_2 c}{c^2 - v^2}$$

$$t_2 = \frac{2l_2}{c}(1 - v^2/c^2)^{-1}$$

$$t_2 = \frac{2l_2}{c}\left(1+\frac{v^2}{c^2}\right)$$

∴ The time difference between the two waves entering the telescope is

$$t_2 - t_1 = \frac{2l_2}{c}\left(1+\frac{v^2}{c^2}\right) - \frac{2l_1}{c}\left(1+\frac{v^2}{2c^2}\right)$$

If the apparatus is rotated through 90°, the arms are interchanged. The time difference is

$$t_2' - t_1' = \frac{2l_2}{c}\left(1+\frac{v^2}{2c^2}\right) - \frac{2l_1}{c}\left(1+\frac{v^2}{c^2}\right)$$

∴ The time shift

$$\Delta t = (t_2' - t_1') - (t_2 - t_1)$$

$$\Delta t = -\left(\frac{l_1 + l_2}{c}\right)\frac{v^2}{c^2}$$

∴ The fringe shift, $\delta = |\upsilon \Delta t|$

$$\delta = \frac{c}{\lambda}\frac{(l_1 + l_2)}{c}\cdot\frac{v^2}{c^2}$$

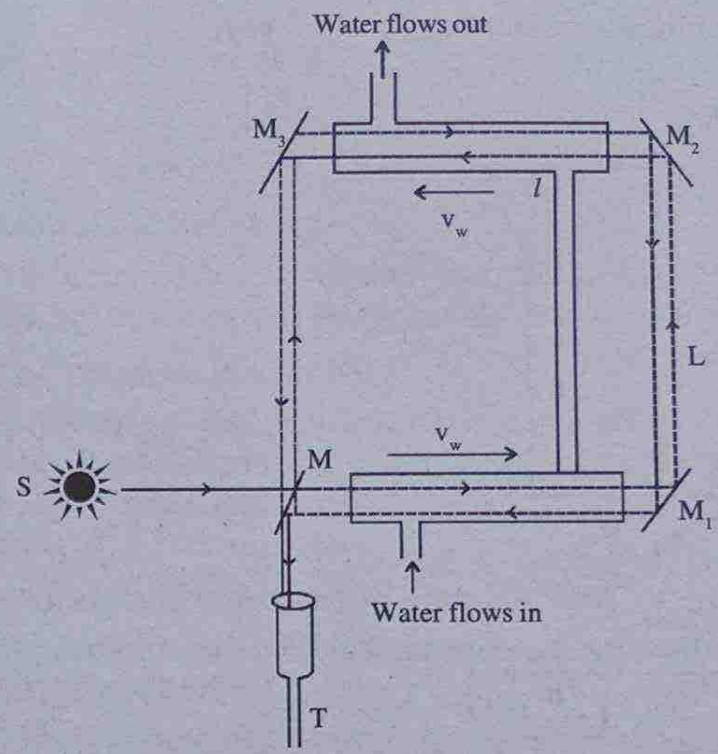
i.e.,

$$\delta = \frac{(l_1 + l_2)}{\lambda}\cdot\frac{v^2}{c^2}$$

Example 3

H. L. Fizeau investigated the velocity of light through a moving medium using the interferometer shown below.

Light of wavelength λ from a source S is split into two beams by the mirror M. The light travel around the interferometer in opposite directions and are combined at the telescope of the observer O who sees the fringe pattern. Two arms of the interferometer pass through water filled tubes of length l with flat glassend plates. The water runs through the tubes, so that one of the light beams travel down stream while the other goes upstream. Calculate the fringe shift.



M, M_1, M_2 and M_3 are mirrors at 45°

Figure 1.2

Solution

Time taken by light to go around the interferometer anticlockwise is

$$t_1 = \frac{l}{v_w + \frac{c}{n}} + \frac{L}{c} + \frac{l}{v_w + \frac{c}{n}} + \frac{L}{c}$$

$$t_1 = \frac{2l}{v_w + \frac{c}{n}} + \frac{2L}{c}$$

where v_w is the speed of water and n is the refractive index of water.

Time taken by light to go around the interferometer clockwise is

$$t_2 = \frac{2l}{\frac{c}{n} - v_w} + \frac{2L}{c}$$

∴ The time difference of light wave entering the telescope is

$$\Delta t = t_2 - t_1 = \frac{2l}{\frac{c}{n} - v_w} - \frac{2l}{v_w + \frac{c}{n}}$$

$$\Delta t = 2l\left(\frac{1}{\frac{c}{n} - v_w} - \frac{1}{v_w + \frac{c}{n}}\right)$$

$$\Delta t = 2l\left(\frac{n}{c - nv_w} - \frac{n}{nv_w + c}\right)$$

$$\Delta t = \frac{2ln}{c}\left(\frac{1}{1 - \frac{nv_w}{c}} - \frac{1}{1 + \frac{nv_w}{c}}\right)$$

$$\Delta t = \frac{2ln}{c}\left[\left(1 - \frac{nv_w}{c}\right)^{-1} - \left(1 + \frac{nv_w}{c}\right)^{-1}\right]$$

$$\Delta t = \frac{2ln}{c}\left[\left(1 + \frac{nv_w}{c}\right) - \left(1 - \frac{nv_w}{c}\right)\right]$$

$$\Delta t = \frac{2ln}{c}\frac{2nv_w}{c}$$

$$\Delta t = 4n^2 l \frac{v_w}{c^2}$$

Hence the fringe shift = $\nu\Delta t = \frac{c}{\lambda}\Delta t$

$$\delta = 4n^2 \frac{l}{\lambda c} v_w$$

Note: The actual fringe measured by Fizeau was $\delta = 4n^2 \frac{l}{\lambda c} v_w f$, where $f = 1 - \frac{1}{n^2}$ known as Fresnel drag coefficient. This was explained only after the advent of relativity.

Postulates of Relativity

The entire edifice of relativity was built upon two basic postulates. They are called (i) The principle of relativity (ii) Principle of constancy of speed of light.

I. The principle of relativity

According to this principle all laws of physics are the same in all inertial frames. It means that it is impossible to designate an inertial frame as stationary or moving by conducting experiments. We can speak of the relative motion of the two frames.

II. Principle of constancy of speed of light

According to this principle the speed of light in free space has the same value in all inertial frames.

These two postulates - one stating that all motion is relative, and the other speed of light is relative to nothing but it is an absolute constant, seem contradictory. But in the world of relativity they do not conflict. No experimental objection to Einsteins special theory of relativity has yet been found.

Galilean transformation

Suppose a physical phenomenon (event) is observed from two separate reference frames, naturally they will have two seperate sets of co-ordinates corresponding to their frames of reference.

Equations relating the two sets of co-ordinates of the event in the two frames are called transformation equations. If the two frames are inertial ones, the transformation is referred to as Galilean transformation.

*[The position and time of occurrence of a physical phenomenon taken together is called event].

Galilean transformation equations

There are three Galilean transformation equations

(i) Galilean co-ordinate transformation.

(ii) Galilean velocity transformation.

(iii) Galilean accelaration transformation.

Galilean coordinate transformation equation

Let us consider two inertial frames of reference S and S'. S be at rest and S' be moving with a uniform velocity v in the positive X-direction. Let t and t' be the time of an event measured by S and S'. Let us consider two observers one in S and other in S'. To begin with let the two frames coincide i.e. $t = t' = 0$. After a time t the two frames observe an event taking place at P. During this time t the S' frame has moved a distance vt along that direction with respect to the rest frame S. With respect to the rest frame S, the co-ordinates are (x, y, z, t) and with respect to the moving frame S', the co-ordinates are (x', y', z', t').

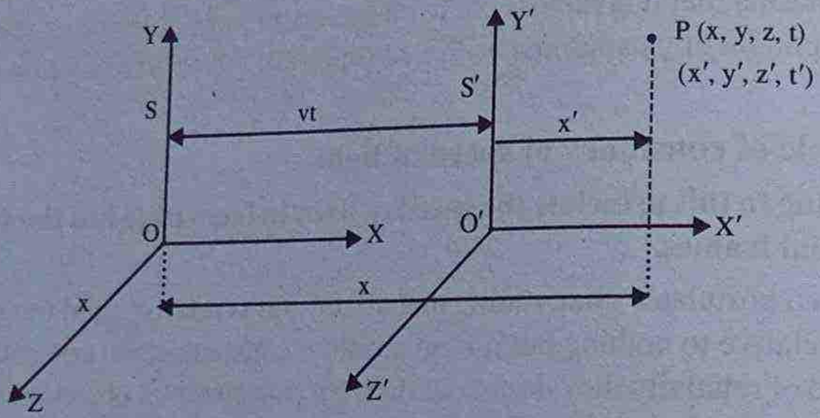


Figure 1.3

Therefore we can write (see figure 1.3)

$$x' = x - vt \qquad \text{...... (1)}$$

Since there is no motion along the Y and Z directions

$$y' = y \quad \text{and} \qquad \text{...... (2)}$$

$$z' = z \qquad \text{...... (3)}$$

Since according to classical idea the event should appear simultaneously to the two observers.

$$t' = t \qquad \text{...... (4)}$$

Equations 1, 2, 3 and 4 are known as Galilean co-ordinate transformation equations.

These transformation equations allow us to go from one inertial frame S to another inertial frame S'. If we substitute this transformation of co-ordinates into Newtons laws we can see that the laws of Newton are of the same form in two frames (see example 2). But if we transform Maxwell's equations by the substitution of above transformations their form does not remain the same. It shows that the above transformation does not give a law of physics equivalant to the two frames of references. That is Galilean transformaltion doesnot satisfy the first postulate of relativity. (see unit III)

Galilean velocity transformation equation

Differentiating equations 1, 2 and 3 with respect to t gives.

$$\frac{dx'}{dt'} = \frac{dx}{dt} - v \quad \left(\frac{d}{dt'} = \frac{d}{dt}, \text{ since } t' = t\right)$$

$$\frac{dy'}{dt'} = \frac{dy}{dt}$$

$$\frac{dz'}{dt'} = \frac{dz}{dt}$$

using $\frac{dx'}{dt'} = u'_x$, the x-component of velocity measured with respect to S′ and $\frac{dx}{dt} = u_x$, the x-component of velocity measured with respect to S and so on. Then the above equations become

$$u'_x = u_x - v \qquad \text{...... (5)}$$

$$u'_y = u_y \qquad \text{...... (6)}$$

$$u'_z = u_z \qquad \text{...... (7)}$$

Equations 5, 6 and 7 are called Galilean velocity transformation equations or simply the classical velocity addition theorem or Galilean law of additon of velocities. Clearly in the more general cases in which velocity v has components along all three axes, we would obtain the general vector result

$$\vec{u}' = \vec{u} - \vec{v} \qquad \text{...... (8)}$$

For example the velocity of an aeroplane with respet to the air $(\vec{u}' = \vec{v}_{PA})$ equals the velocity of the plane with respect to the ground $(\vec{u} = \vec{v}_{PG})$ minus the velocity of the air with respect to the ground $(\vec{v} = \vec{v}_{AG})$

i.e., $\vec{v}_{PA} = \vec{v}_{PG} - \vec{v}_{AG}$

or $\vec{v}_{PA} + \vec{v}_{AG} = \vec{v}_{PG}$

See the same subscript symbols fusing to get the result.

If we apply these Galilean velocity transformation equations to particles which are moving very fast, we can see that this equations fail badly. For example let $u'_x = c'$ and $u_x = c$ be the velocity of light observed by the persons in S' and S respectively

i.e., $c' = c - v$

This equation shows that velocity of light is different for different inertial frames. It is contrary to experimentally observed fact that velocity of light is same for all inertial frames. That is it violates the second postulates of relativity (see unit III).

Galilean acceleration transformation equations

Differenting equations 5, 6 and 7 with respect to $t(t = t')$

we get $\frac{du'_x}{dt'} = \frac{du_x}{dt} \quad \left(\because \frac{dv}{dt} = 0\right)$

The second term on the R.H.S of equation 5 does not give a derivative with respect to time since $\vec{v}$ the velocity with which the frame S' is moving is a constant.

$$\frac{du'_y}{dt'} = \frac{du_y}{dt}$$

and $\frac{du'_z}{dt'} = \frac{du_z}{dt}$

$\frac{du'_x}{dt'} = a'_x$, the accelеartion of the event with respect to the S' frame and $\frac{du_x}{dt} = a_x$, the acceleration of the event with respect to the S frame and so on.

Thus we have

$$a'_x = a_x \quad \text{...... (9)}$$

$$a'_y = a_y \quad \text{...... (10)}$$

$$a'_z = a_z \quad \text{...... (11)}$$

It shows that both frames (or observers) measure the same acceleration. Thus according to Galilean transformation, the acceleration of a particle is the same for all observers (frames) in uniform relative motion. Since the acceleration remains invariant under Galilean transformations [passing from one inertial frame (rest) to another inertial frame (uniform motion)], acceleration is called Galilean invariant physcal quantity.

In general physical quantities which remain invariant under Galilean transformation equations are called Galilean invariant quantities.

Example 4

A rocket travelling at a speed of 500 m/s ejects the burnt gases in opposite direction from its rear. If the speed of the ejected gases relative to the ground is 1000m/s. Find the speed of the ejected gases with respect to the rocket.

Solution

Take the ground to be the rest frame (S), moving rocket to be the S' frame and the event served to be the ejected gas.

Velocity of the ejected gas with respect to ground, $u_x = -1000$ m/s

Velocity of the rocket with respect to the ground, $v = 500$ m/s

Velocity of ejected gas with respect to the rocket?

$$\begin{aligned} u'_x &= u_x - v \\ &= -1000 - 500 \\ &= -1500 \text{ ms}^{-1} \end{aligned}$$

Example 5

Show that the force acting on a particle as observed by two observers in two inertial frames of reference is the same under Galilean transformation (i.e. $v \ll c$)

Solution

Consider two inertial frames S and S'. S be at rest and S' be moving with a velocity v with respect to S along the positive x-direction. Let us consider two observers one in S and another in S'. Consider a particle of mass m moving with a velocity u_1 with respect to S. Let a force F act on the particle for a time t which changes its velocity to u_2.

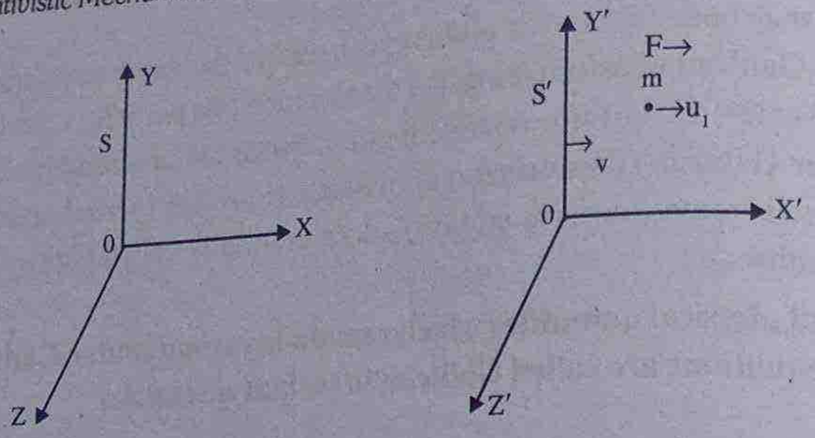


Figure : 1.4

Example 6

Two light pulses are emitted in the opposite directions from a source at rest. What is the speed of one light pulse as measured from the other.

Solution

Let the rest frame S be attached to the source and the moving frame S′ be attached with the pulse moving along the x-direction. The S′ frame is observing the light pulse moving along the –x direction.

Then

velocity of the light pulse with respect to S, $u_x = -c$

–ve sign comes, since the observed light pulse is moving in the –x direction

velocity of the S′ frame v = c

∴ The velocity of the observed light pulse with respect to S′,

$$u'_x = u_x - v$$
$$u'_x = -c - c = -2c$$

–ve sign shows that light pulse is moving along the –ve x-direction. It shows that the observed velocity of the light pulse is different in different frames, thus violating the second postulate of relativity. This tells us that Galilean transformation equations are not suitable to apply to fast moving particles ($v \to c$)

Example 7

Show that the length or distance between two points is invariant under Galilean transformation.

Solution

Let S and S′ be two frames. S be at rest and S′ be moving with a speed v along the +ve x-direction. Consider a rod of length L in the frame S with co-ordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) with respect to S′ frame the co-ordinates are (x_1', y_1', z_1') and (x_2', y_2', z_2').

The length of the rod with respect to S frame

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \qquad \text{...... (1)}$$

The length of the rod with respect to S′ frame

$$L' = \sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2} \qquad \text{...... (2)}$$

using $x' = x - vt$

we have $x_1' = x_1 - vt$

and $x_2' = x_2 - vt$

$\therefore \quad x_2' - x_1' = x_2 - x_1$

using $y' = y$

$y_1' = y_1$ and $y_2' = y_2$

Then $y_2' - y_1' = y_2 - y_1$

using $z' = z$

$z_1' = z_1$

$z_2' = z_2$

$\therefore \quad z_2' - z_1' = z_2 - z_1$

Substituting these in eqn. 2, we get

$$L' = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Thus $L' = L$ Hence the result

Lorentz transformation equations

We have seen that Galilean transformation equations do not obey first postulate and second postulate of relativity (see unit - I). Therefore Galiliean transformation equations must be replaced by new ones consistent with experiment. These new equations are called Lorentz transformation equations.

Consider two inertial frames S and S′. Let S be at rest and S′ be moving with a uniform velocity v in the positive x-direction. Let t and t' be the time of an event measured by S and S' respectively. To begin with let the two frames coincide i.e. $t=0, t'=0$. After a time t (with respect to S) an event taking place at P represented by (x, y, z, t). The same event is represented by S' as (x', y', z', t'). After a time t S' frame has moved forward to a distance vt with respect to the S frame.

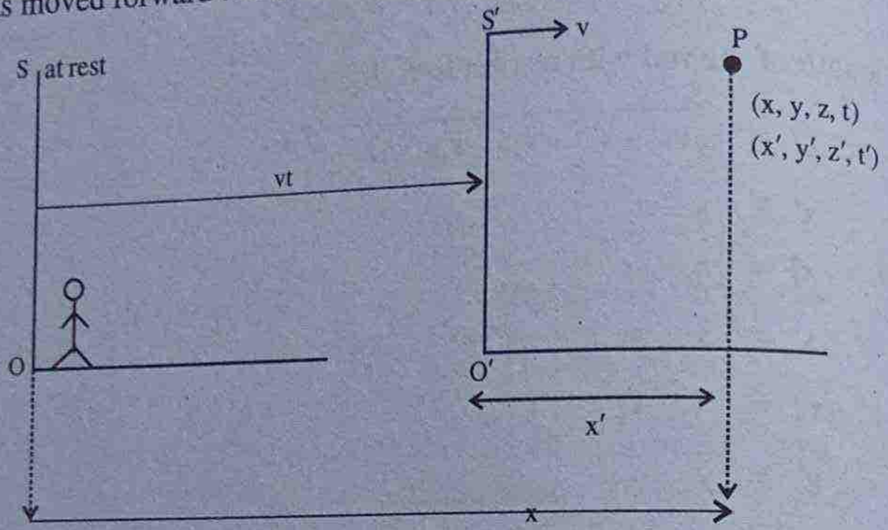


Figure 1.5

The simplest possible form of this equation can be

$$x' = k(x - vt) \qquad \text{...... (7)}$$

This is a linear equation. It should be so since a single event should appear as single to both the observers. The constant k is used in equation (7) to satisfy the two postulates of relativity. It will be a function of v, since the difference is caused by the relative motion between the two frames. According to the first postulate the observations made in the S frame must be identical to those made in S' frame except for a change in the sign of v and having the same value of k.

$$x = k(x' + vt') \qquad \text{...... (8)}$$

Here consider S' be at rest and S be moving backward with a velocity v, also the time measured by the S' frame is t'. Therefore the distance between the two frames will appear as vt'.

Since the motion is confined to x-direction only

$$y' = y \qquad \text{...... (9)}$$

$$z' = z \qquad \text{...... (10)}$$

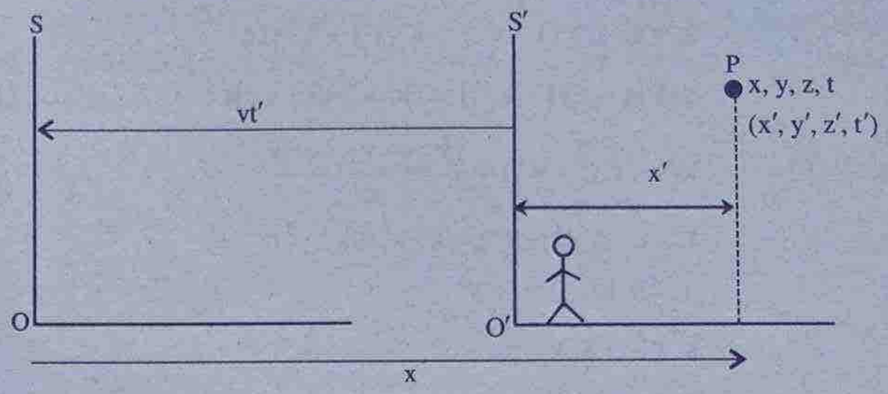


Figure 1.6

To find t'

Substitute for x' from equation (7) in equation (8)

$$x = k\{k(x - vt) + vt'\}$$

$$x = k^2x - k^2vt + kvt'$$

$$kvt' = x - k^2x + k^2vt$$

$$\therefore \quad t' = \frac{x(1-k^2)}{kv} + kt \qquad \text{...... (11)}$$

To find k

Consider again two frames S and S'. The observer in the S frame measures the time t and the observer in the S' frame measures the time t' for a flash of light. The observer in the S frame sees that the flash has moved through a distance

$$x = ct \qquad \text{...... (12)}$$

and the observer in the S' frame sees that the flash has moved through a distance

$$x' = ct' \qquad \text{...... (13)}$$

The velocity of light is assumed to be the same in both frames according to second postulate.

Substituting for x' and t' in equation (13) from equations (7) and (11) respectively, we get

$$k(x - vt) = c\left\{\frac{x(1-k^2) + k^2vt}{kv}\right\}$$

$$k^2v(x - vt) = c\,x(1-k^2) + k^2vtc$$

$$k^2vx - k^2v^2t = c\,x(1-k^2) + k^2vtc$$

$$k^2vx - c\,x(1-k^2) = k^2\,v^2t + k^2\,vtc$$

$$x[k^2v - c\,(1-k^2)] = (k^2\,v^2 + k^2\,vc)t \qquad \text{...... (14)}$$

$\frac{\text{equation 14}}{\text{equation 12}}$ gives

$$k^2v - c\,(1-k^2) = \frac{(k^2\,v^2 + k^2\,vc)}{c}$$

$$k^2vc - c^2(1-k^2) = k^2\,v^2 + k^2\,v^2\,c$$

$$-c^2 + k^2\,c^2 = k^2v^2$$

$$k^2\,c^2 - k^2v^2 = c^2$$

$$k^2\,(c^2 - v^2) = c^2$$

$$k^2 = \frac{c^2}{c^2 - v^2} = \frac{1}{1 - v^2/c^2}$$

$$k = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Substituting the value of k in equations (7), (9), (10) and (11), we get

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \qquad \text{...... (15)}$$

$$y' = y \qquad \text{...... (16)}$$

$$z' = z \qquad \text{...... (17)}$$

$$t' = \frac{x\left(1 - \frac{1}{1 - v^2/c^2}\right)}{\frac{1}{\sqrt{1 - v^2/c^2}}\,v} + \frac{t}{\sqrt{1 - v^2/c^2}}$$

$$t' = \frac{x\left(1 - \frac{v^2}{c^2} - 1\right)}{\frac{v}{\sqrt{1 - v^2/c^2}}\,(1 - v^2/c^2)} + \frac{t}{\sqrt{1 - v^2/c^2}}$$

$$t' = \frac{-x\,v^2/c^2}{v\sqrt{1 - v^2/c^2}} + \frac{t}{\sqrt{1 - v^2/c^2}}$$

$$t' = \frac{-x\,v/c^2}{\sqrt{1 - v^2/c^2}} + \frac{t}{\sqrt{1 - v^2/c^2}}$$

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \qquad \text{...... (18)}$$

Equations (15), (16), (17) and (18) are called Lorentz transformation equations. The above equations can be written in another form

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \quad \text{...... (19)}$$

$$y = y' \quad \text{...... (20)}$$

$$z = z' \quad \text{...... (21)}$$

$$t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}} \quad \text{...... (22)}$$

Where v of the former equations are replaced by -v and we interchanged primed and unprimed quantities. Equations 19, 20, 21 and 22 are called the inverse Lorentz transformation equations.

It is now clear that the measurement of space and time are by no means absolute but are dependant upon the relative motion between the observer and the phenomenon observed. When $\frac{v}{c} \ll 1$, the Lorentz transformation equations reduce to the Galilean transformation equations.

Example 8

At what speed v, will the Galilean and Lorentz expressions for x differ by 10%.

Solution

$$x_G = x - vt$$

$$x_L = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$\therefore \quad \frac{x_L - x_G}{x_G} = \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right)$$

But $$\frac{x_L - x_G}{x_G} = 10\% = \frac{10}{100} = \frac{1}{10} = 0\cdot 1$$

$$0\cdot 1 = \frac{1}{\sqrt{1 - v^2/c^2}} - 1$$

$$1\cdot 1 = \frac{1}{\sqrt{1 - v^2/c^2}} \quad \text{squaring on both sides}$$

$$1\cdot 21 = \frac{1}{1 - v^2/c^2}$$

$$1-\frac{v^2}{c^2}=\frac{1}{1\cdot 21}$$

$$\frac{v^2}{c^2}=1-\frac{1}{1\cdot 21}=\frac{0\cdot 21}{1\cdot 21}=\frac{21}{121}$$

$$\frac{v}{c}=0\cdot 417 \quad \text{or} \quad v=0\cdot 417c$$

See the enormous speed required to have a change by 10%.

Relativistic kinematics

We found that classical mechanics obeys the Galilean transformation whereas in relativity it obeys Lorentz transformations. This shows that in the realm of relativity all Newtonian results need modification, there we develop the kinematics appropriate to the Lorentz transformation. The Lorentz transformation equations and their inverse transformation are

$$\left.\begin{aligned} x' &= \gamma(x-vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t-\frac{vx}{c^2}\right), \text{ where } \gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \end{aligned}\right] \text{Lorentz transformation}$$

$$\left.\begin{aligned} x &= \gamma(x'\times vt') \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t'+\frac{vx'}{c^2}\right) \end{aligned}\right] \text{Inverse Lorentz transformation equations}$$

The Lorentz transformation equations show that the co-ordinates of the event(x, y, z, t) with respect to one inertial frame (S) is not equal to the co-ordinates of the same event (x',y',z',t') with respect to another inertial frame (S') (see example 8 and 9). From this we can very well conclude that space and time are relative. That is the measurements of length (space) and time depend upon the nature of the inertial frames. One more thing to be noted here is that going from Newtonian mechanics to relativity the main change is brought by the factor γ. The variation of γ with $\frac{v}{c}$ is shown

In relativity the limiting speed is velocity of light where as in Newtonian mechanics in principle the limiting speed is ∞. So if you replace c by ∞ in Lorentz transformation equations we get back Galilean transformation equations.

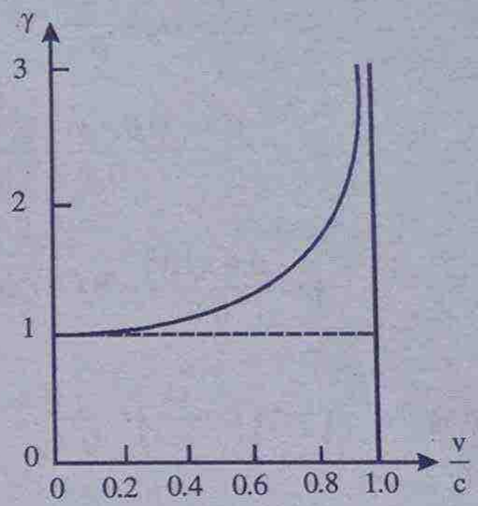


Figure 1.7

Example 9

Consider two inertial frames S (x, y, z, t) and $S'(x', y', z', t')$. S' is moving with speed v relative to S along the x-axis. The origins at $t = t' = 0$

Assume that $v = 0.6c$ find the co-ordinates in S' of the following.

(a) $x = 4m$, $t = 0$ s (b) $x = 10^9 m$, $t = 2s$

Solution

We have $x' = \gamma(x - vt)$

$$\therefore \quad x' = \gamma(4 - 0) = \gamma 4$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - 0.6^2 \frac{c^2}{c^2}}} = \frac{1}{\sqrt{0.64}} = \frac{1}{0.8}$$

$$\therefore \quad x' = \frac{4}{0.8} = 5m$$

Using $$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$t' = \frac{1}{0.8}\left(0 - \frac{0.6c \times 4}{c^2}\right)$$

$$t' = -\frac{0.6 \times 4}{0.8 \times c} = -\frac{2.4}{0.8 \times 3 \times 10^8}$$

$$t' = -10^{-8} s$$

b) $$x' = \gamma(x - vt) = \frac{1}{0.8}(10^9 - 0.6c \times 2)$$

$$x' = \left(\frac{10^9 - 0.6 \times 3 \times 10^8 \times 2}{0.8}\right)$$

$$x' = \frac{6.4 \times 10^8}{0.8} = 8 \times 10^8 \text{m}$$

Using $$t' = \gamma\left(t - \frac{vx}{c^2}\right) = \frac{1}{0.8}\left(2 - \frac{0.6c \times 10^9}{c^2}\right)$$

$$t' = \frac{1}{0.8}\left(2 - \frac{0.6 \times 10^9}{3 \times 10^8}\right) = \frac{1}{0.8}(2 - 2)$$

$$t' = 0$$

Example 10

An event occurs in S at $x = 6 \times 10^8$ m and in S′ at $x' = 6 \times 10^8$ m, $t' = 4$s Find the relative velocity of the systems.

Solution

$x = 6 \times 10^8$ m, $x' = 6 \times 10^8$ m and $t' = 4$s

Using inverse Lorentz transformation

$$x = \gamma(x' + vt')$$

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}$$

$$6 \times 10^8 = \frac{6 \times 10^8 + v \times 4}{\sqrt{1 - v^2/c^2}}$$

or $$\sqrt{1 - v^2/c^2} = \frac{6 \times 10^8 + 4v}{6 \times 10^8}$$

$$\sqrt{1 - v^2/c^2} = 1 + \frac{4v}{6 \times 10^8} = 1 + 2\frac{v}{c}$$

Squaring on both sides we get

$$1-\frac{v^2}{c^2}=1+4\frac{v}{c}+\frac{4v^2}{c^2}$$

$$-\frac{v^2}{c^2}=4\frac{v}{c}+4\frac{v^2}{c^2}$$

$$-5\frac{v^2}{c^2}=4\frac{v}{c}$$

$$-5\frac{v}{c}=4$$

$$v=-\frac{4}{5}c=-\frac{4\times 3\times 10^8}{5}$$

$$v=-2.4\times 10^8\,ms^{-1}$$

Simultaneity

We have seen that space and time are relative in its nature. One of the most important consequences of this nature is that simultaneity is relative. The events that seem to take place simultaneously to one observer may not be simultaneous to another observer in relative motion and vice versa.

Suppose that two events take place at same instant at two different positions x_1 and x_2 in the frame of reference S. An observer in the frames S' moving relative to S would measure the instants at which the two events occur as

$$t_1'=\frac{t_1-\frac{vx_1}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{...... (1)}$$

and

$$t_2'=\frac{t_2-\frac{vx_2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{...... (2)}$$

eq (2) - eq (1) gives

$$t_2' - t_1' = \frac{(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1)}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$t_2 - t_1$ is the time interval measured in the S frame. Since the events take place simultaneously with respect to S frame $t_2 - t_1 = 0$.

$$\therefore \quad t_2' - t_1' = \frac{\frac{v}{c^2}(x_1 - x_2)}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This shows that $t_2' - t_1'$, which is the time intervals of two events with respect to S' frame is not zero. Thus the observer in the S' frame concludes that events are not simultaneous. i.e. The two events that are simultaneous in one frame are not simultaneous in another frame of reference.

In other words simultaneity of events is relative.

(see examples 9 and 10)

The order of events: Timelike and space like intervals

Consider two events A and B occur in space. An observer in the S frame measures the co-ordinates of A and B as (x_A, t_A) and (x_B, t_B).

With respect to the S frame, the events are separated by a distance say L

$$L = x_B - x_A$$

and the time separation of the events be

$$T = t_B - t_A$$

Assume that $x_B > x_A$ and $t_B > t_A$, so that L and T are positive.

Suppose these events A and B are observed by observer in the S' frame moving with a speed v along the positive x-axis. Let x_A', t_A' be the coordinates of event A and x_B', t_B' be the coordinates of event B with respect to S' frame.

The space separation between the two events A and B with respect to S' frame is

$$L' = x_B' - x_A'$$

Using Lorentz transformation equation

$$x' = \gamma(x - vt)$$

we get $\quad x'_B = \gamma(x_B - vt_B)$

and $\quad x'_A = \gamma(x_A - vt_A)$

$\therefore \quad L' = \gamma[(x_B - x_A) - v(t_B - t_A)]$

But $\quad x_B - x_A = L$ and $t_B - t_A = T$

Thus, $\quad L' = \gamma(L - vT)$ .....(1)

Similarly the time interval between two events A and B with respect to S' frame is

$$T' = t'_B - t'_A$$

Using $\quad t' = \gamma(t - vx/c^2)$

we have $\quad t'_B = \gamma\left(t_B - v\frac{x_B}{c^2}\right)$

and $\quad t'_A = \gamma\left(t_A - v\frac{x_A}{c^2}\right)$

$\therefore \quad T' = \gamma\left[(t_B - t_{A)} - \frac{v}{c^2}(x_B - x_A)\right]$

or $\quad T' = \gamma\left(T - \frac{v}{c^2}L\right)$ .....(2)

If $L > cT$, equation (1) shows that

L' is always positive

If $L > cT$ equation (2) becomes

$$T' = \gamma\left(T - \frac{v}{c^2} \times > cT\right)$$

or $\quad T' = \gamma\left(T - > \frac{v}{c}T\right)$

Depending upon the value of v, T' can be positive, zero or negative.

When the space intervals is positive and the time interval is positive, zero or negative, then the space-time interval is called space like. For this to happen L' must be greater than cT. In a space like interval it is possible to choose an inertial system in which the events are simultaneous. Events in S' frame simultaneous means $T' = 0$. This gives $T = \frac{v}{c^2}L$ or $v = c^2\frac{T}{L}$.

If $L < cT$, equation (1) shows that L' can be positive, negative or zero. But equation (2) shows that, T' is always positive. That is when the time interval is positive and the space interval is positive, zero or negative, then the space-time interval is said to be time like. In a time like interval it is always possible to find an inertial system in which the events occur at the same point.

Lorentz length contraction

The length of a body is measured to be greatest when it is at rest relative to the observer. When it moves with a velocity relative to the observer its length is contracted in the direction of its motion where as its dimension perpendicular to the direction are unaffected.

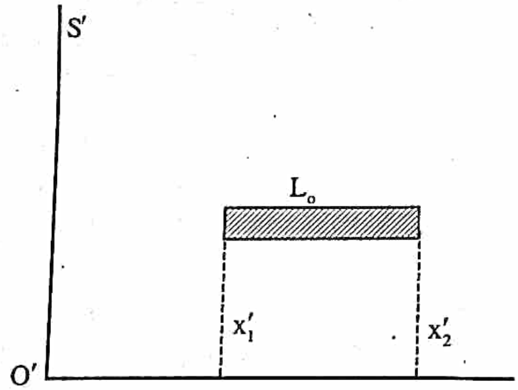


Figure 1.8

Proof

Consider a rod lying at rest along the x'– axis of the S' frame which moves with a velocity v in the positive x- direction. Its end points are measured to be at x_2' and x_1' so that its length.

$$L_0 = x_2' - x_1' \qquad \text{...... (1)}$$

An observer in the rest frame measures the points as x_2 and x_1. The length L as measured by the observer in S is

$$L = x_2 - x_1 \qquad \text{...... (2)}$$

From the first Lorentz transformation equations we have

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$\therefore \qquad x_1' = \frac{x_1 - vt_1}{\sqrt{1 - v^2/c^2}}$$

$$x_2' = \frac{x_2 - vt_2}{\sqrt{1 - v^2/c^2}}$$

So that
$$x_2' - x_1' = \frac{(x_2 - x_1) - v(t_2 - t_1)}{\sqrt{1 - v^2/c^2}}$$

But the measurement of length involves the simultaneous determination of the spatial coordinates of its end point whether it is with respect to an observer at rest or in motion i.e., $t_2 = t_1$.

$$\therefore \qquad x_2' - x_1' = \frac{x_2 - x_1}{\sqrt{1 - v^2/c^2}}$$

Substituting for $x_2' - x_1'$ and $x_2 - x_1$ from equations (1) and (2) we have

$$L_0 = \frac{L}{\sqrt{1 - v^2/c^2}}$$

or
$$L = L_0 \sqrt{1 - v^2/c^2}$$

i.e.,
$$L < L_0.$$

This shows that the length of a moving rod appears to contract from its rest length in the direction of its motion. This is called Lorentz - Fitzgerald contraction. This contraction is appreciable only when the velocity v is comparable to the velocity of light c. Even then it has been verified experimentally several times.

When $v = c, L = o$

This shows that it is impossible to impart a velocity equal to that of light to a body. It is due to length contraction a fastly moving ring appears to be oblate in shape (ellipse), sphere appears to be an ellipsoid, a cube appears to be a parallelopiped and so on.

Example 11

The length of a space ship is measured to be exactly half its proper length. What is the speed of the space ship relative to the observers frame.

Solution

We have

$$L = L_0 \sqrt{1 - v^2/c^2}$$

$$L = \frac{L_0}{2} \text{ given}$$

$$\therefore \quad \frac{L_0}{2} = L_0 \sqrt{1 - v^2/c^2}$$

$$\tfrac{1}{2} = \sqrt{1 - v^2/c^2} \text{ Squaring on both sides}$$

$$\tfrac{1}{4} = 1 - \frac{v^2}{c^2}$$

$$\frac{v^2}{c^2} = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\frac{v}{c} = \frac{\sqrt{3}}{2} = 0 \cdot 866$$

$$v = 0 \cdot 866\,c = 2 \cdot 598 \times 10^8\,\text{ms}^{-1}$$

Example 12

A circular ring in $x - y$ plane moves parallel to the x- axis. What should be its velocity so that its area appears to be half the stationary area.

Solution

Area of the ring $= \pi R_0^2, R_0$ be the radius of the ring. As it moves along x-direction, its radius along x-axis suffers contraction. As a result the ring assumes the shape of an ellipse with semi major axis R_0 and semi minor axis R

Area of the ellipse $= \pi R_0 R$

As this area appears to be half that of ring we have

$$\pi R_0 R = \frac{\pi R_0^2}{2} \qquad R = \frac{R_0}{2}$$

using $\quad L = L_0 \sqrt{1 - v^2/c^2}$ with $L_0 = R_0, L = R = \frac{R_0}{2}$

$$\frac{R_0}{2} = R_0\sqrt{1 - v^2/c^2}$$

$\frac{1}{2} = \sqrt{1 - v^2/c^2}$ squaring on both sides

$$\frac{1}{4} = 1 - v^2/c^2$$

$$\frac{v^2}{c^2} = 1 - \frac{1}{4} = \frac{3}{4}$$

$$v = \frac{\sqrt{8}}{2}c = 0{\cdot}866\ c = 2{\cdot}598 \times 10^8\,ms^{-1}$$

The orientation of a moving rod

Consider a rod of length L' lies in the S' frame making an angle θ'. The frame S' is moving with a velocity v along the positive x-direction. Here our aim is to calculate the length and orientation of the rod with respect to an observer in the S frame.

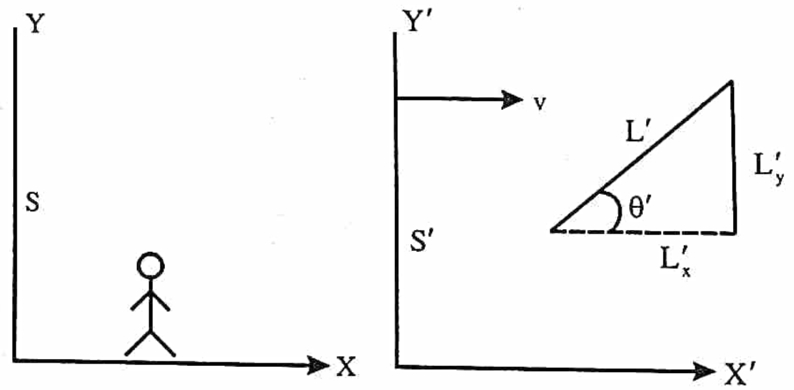


Figure 1.9

From the S' frame, we have

$$L'_x = L'\cos\theta'$$

and $$L'_y = L'\sin\theta'$$

$$\therefore \quad \frac{L'_y}{L'_x} = \tan\theta' \qquad \text{......(1)}$$

Let L be the length of the rod and θ be the orientation of the rod with respect to the S frame. Then

$$L_x = L\cos\theta$$

and $$L_y = L\sin\theta$$

$$\therefore \quad \frac{L_y}{L_x} = \tan\theta \qquad ....(2)$$

Using $L = L_0\sqrt{1-v^2/c^2}$

$$L_x = L'_x\sqrt{1-v^2/c^2}$$

and $L_y = L'_y$ (no contraction along y-direction)

$$\therefore \quad \frac{L_y}{L_x} = \gamma\frac{L'_y}{L'_x}$$

substituting for $\frac{L_y}{L_x}$ and $\frac{L'_y}{L'_x}$ from equations 1 and 2 we get

$$\tan\theta = \gamma\tan\theta'$$

or $$\theta = \tan(\gamma\tan\theta') \qquad .....(3)$$

To find the length of the rod, we use Pythagorus theorem

$$L^2 = L_x^2 + L_y^2$$

$$L^2 = \left(L'_x\sqrt{1-\frac{v^2}{c^2}}\right)^2 + L'^2_y$$

or $$L^2 = (L'\cos\theta'\sqrt{1-v^2/c^2})^2 + (L'\sin\theta')^2$$

$$L^2 = L'^2\cos^2\theta'\left(1-\frac{v^2}{c^2}\right) + L'^2\sin^2\theta'$$

$$L^2 = L'^2\left(1-\frac{v^2}{c^2}\cos^2\theta'\right)$$

$$L = L'\left(1 - \frac{v^2}{c^2}\cos^2\theta'\right)^{\frac{1}{2}} \qquad .....(4)$$

Equation 3 says that the angle measured by an observer in the S frame appears to increase and equation 4 says that length appears to be shortened with respect to S frame observer. See examples 13 and 14.

Example 13

A space craft antenna is at an angle of 10° relative to the axis of the space craft. If the space craft moves away with a speed of 0.7c. What is the angle of the antenna as seen from the earth.

Solution

$$\theta' = 10^\circ$$
$$v = 0.7c$$

Using $\theta = \tan^{-1}(\gamma \tan\theta')$

$$v = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - 0.7^2}} = \frac{1}{\sqrt{.51}} = 1.4003$$

$$\theta = \tan^{-1}(1.4003 \tan 10^\circ)$$

$$\theta = \tan^{-1}(0.2469) = 13.86^\circ$$

Example 14

A light beam is emitted at angle θ' with respect to the x'-axis in S'. Find the angle θ the beam makes with respect to the x-axis in S

Solution

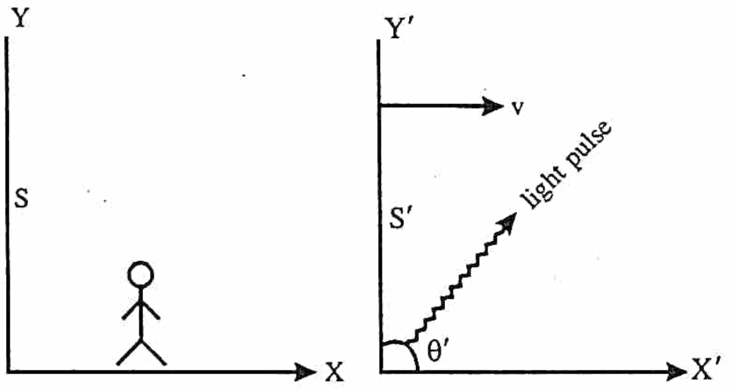


Figure 1.10

Assume that the light wave starts from the origin O'. After a time t', the x', y' coordinates of light pulse are

$$x' = ct' \cos\theta'$$

and $$y' = ct' \sin\theta'$$

with respect to the S frame the coordinates are

$$x = \gamma(x' + vt') = \gamma t'c\left(\cos\theta' + \frac{v}{c}\right)$$

$$y = y'$$

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

$$t = \gamma t'\left(1 + \frac{v}{c}\cos\theta'\right)$$

From the S frame we also have

$$x = ct\ \cos\theta$$

$$\therefore \qquad \cos\theta = \frac{x}{ct} = \frac{\gamma t'c\left(\cos\theta' + \frac{v}{c}\right)}{cvt'\left(1 + \frac{v}{c}\cos\theta'\right)}$$

or $$\theta = \cos^{-1}\left[\frac{\left(\cos\theta' + \frac{v}{c}\right)}{\left(1 + \frac{v}{c}\cos\theta'\right)}\right]$$

Time dilation

According to relativity time intervals are affected by the relative motion between frames. Consider two inertial frames S and S'. Let S be at rest and S' be moving with a velocity v in the positive x-direction. To begin with let the two frames coincide i.e. $t = 0, t' = 0$. Then two events occur at any given point x' in frame S' at times t_1' and t_2' as noted on the clock carried by it and times t_1 and t_2 as noted on the colck carried by frame S. That is the time interval between two events as noted on the clock in the moving frame S' is $t_2' - t_1' = \Delta t_0$ and time interval between same two events as noted on the clock in the rest frame S is $t_2 - t_1 = \Delta t$.

Using Lorentz inverse transformation

$$t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$$

$$t_1 = \frac{t_1' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$$

$$t_2 = \frac{t_2' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$$

$$t_2 - t_1 = \frac{t_2' - t_1'}{\sqrt{1 - v^2/c^2}}$$

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}}$$

i.e., $\Delta t > \Delta t'$.

That is an interval of time observed in the rest frame is longer than in the S' frame. This effect is called time dilation. This means that a moving clock runs slow with respect to a stationary clock. Remember that smaller time intervals means moving slow.

When $v \to c$, $\Delta t' \to 0$

That is, the passage of time and also the process of aging will be stopped. This explains what has come to be known as the twin paradox.

Example 15

A particle with mean proper life time of 2×10^{-6} s moves through the laboratory with a speed of 0.99 c. Calculate its life time as measured by an observer in laboratory.

Solution

$$\Delta t' = 2 \times 10^{-6} \text{s}$$

$$v = 0 \cdot 998 c$$

Using $$\Delta t = \frac{\Delta t'}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2\times10^{-6}}{\sqrt{1-9\cdot98^2}} \simeq 32\times10^{-6}\text{s}.$$

Example 16

Consider two identical twins of age 25 years. One remains on earth the other travels within a space ship with a velocity $\frac{\sqrt{3}}{2}$c. After 25 years elapsed on earth, traveller returns. Then what are their ages?

Solution

$$v = \frac{\sqrt{3}}{2}c$$

$$\Delta t = 25 \text{ years}$$

$$\Delta t' = ?$$

Using $$\Delta t = \frac{\Delta t'}{\sqrt{1-v^2/c^2}}$$

$$\Delta t' = \Delta t\sqrt{1-v^2/c^2} = 25\sqrt{1-\frac{3}{4}} = 12\cdot5\text{ years}.$$

$\therefore$ Age of the traveller $= 25+12\cdot5 = 37\cdot5$ years.

Age of one who stayed on earth $= 25+25 = 50$ years.

Example 17

A particle with a mean proper life of 1 μs moves through the laboratory at $2\cdot7\times10^8\text{ms}^{-1}$. What will be the distance travelled by it before disintegration.

Solution

$$v = 2\cdot7\times10^8\text{ms}^{-1}$$

$$\Delta t' = 10^{-6}\text{s}$$

$$\Delta t = \frac{\Delta t'}{\sqrt{1-v^2/c^2}} = \frac{10^{-6}}{\sqrt{1-\left(\frac{2\cdot7\times10^8}{3\times109}\right)^2}} = 2\cdot3\times10^{-6}\text{s}$$

$\therefore$ Distance travelled $x = v\Delta t = 2\cdot7\times10^8\times2\cdot3\times10^{-6} = 620\text{ m}$

Proper frame, proper length and proper time

The inertial frame of reference in which observed body is at rest is called the proper frame of reference. **The length of a rod as measured in the inertial frame in which it is at rest is called the proper length.**

In the relation $L = L_0 \sqrt{1 - v^2/c^2}$

L_0 is called the proper length.

Like wise the proper time interval is the time interval recorded by a clock attached to the observed body. The relation between the proper time interval $\Delta\tau_0$ and non proper time $(\Delta\tau)$ is as follows.

$$\Delta\tau = \frac{\Delta\tau_0}{\sqrt{1 - v^2/c^2}}.$$

Proper time interval is an invariant quantity in relativity.

Muon decay

The time dilation effect can be indirectly verified by the fact of μ^{+} mesons reaching the ground level. These elementary particles are produced in the upper atmosphere as a result of collision of cosmic rays with the air molecules. Though they move with different velocities the faster ones among them have a velocity of 0.998 c.

These particles can travel a distance of

$$x = vt = 0 \cdot 998c \times 2 \times 10^{-6} = 600 \text{ m}$$

where $t = 2 \times 10^{-6}$s is the deacy time of μ mesons. These particles are produced at an altitude of more than 10km above the surface of the earth can hardly reach the surface of earth. But a large number of mesons reach the earths surface. This is because of time dilation. The decay time 2×10^{-6}s is the time with respect to their own frame of reference. i.e. with respect to a person on the earths surface this time will be dilated. Using equation

$$\Delta\tau = \frac{\Delta\tau_0}{\sqrt{1 - v^2/c^2}} = \frac{2 \times 10^{-6}}{\sqrt{1 - (0 \cdot 998)^2}} = 32 \times 10^{-6}\text{s}$$

i.e. more than 16 times than in their own frame of reference. In this much longer life time, they can travel a distance

$$x = vt = 0 \cdot 998 \times 3 \times 10^{8} \times 32 \times 10^{-6}$$

$$= 10 \text{ km}$$

This explains the presence of mesons near the surface of the earth and indirectly verifies the time dilation effect.

Role of time dilation in atomic clock

Now a days atomic clocks are used as standard references to measure unit of length(metre) and also unit of time (second). This is because atomic clocks have several advantages. They are easily accessible, invariable and highly precise. Here we talk about the measurement time and time dilation effect.

When an atom jump from an higher state to lower state it emits radiation with definite frequency given by $\upsilon \simeq \frac{\Delta E}{h} = \frac{1.6 \times 10^{-19}}{6.6 \times 10^{-34}} \simeq 10^{15}\,\text{Hz}$. If ΔE is of the order of electron volt, light emitted is in the optical region.

When the energy change is very small the emitted radiation is in the microwave region $(\upsilon \simeq 10^{10}\,\text{Hz})$.

These microwave radiations can be detected and amplified electronically and need as our standard reference to govern the rate of an atomic clock. Atomic clocks are highly precise. Then precision is of the order of I part 10^{13}. This means that, for example, two maser clocks run for 33,000,00 years, they commit an error of 1 second.

Each atom radiating at its natural frequency serves as a miniature clock. Atoms in a gas are in random motion so these clocks are not at rest with respect to the rest (Laboratory) frame. As a result the observed frequency as shifted due to time dilation effect. Hence we calculate the effect of time dilation.

Consider an atom emitting radiation of its natural frequency υ_0 in its frame. The observed frequency in the lab frame is υ.

Using $\upsilon_0 = \frac{1}{\Delta t_0}$

and $\upsilon = \frac{1}{\Delta t}$

or $\frac{\upsilon}{\upsilon_0} = \frac{\Delta t_0}{\Delta t}$

Using $\Delta t = \gamma \Delta t_0$, we get

$$\frac{\upsilon}{\upsilon_0} = \frac{1}{\gamma}$$

or $$\upsilon = \frac{1}{\gamma}\upsilon_0$$

$$\upsilon = \upsilon_0\sqrt{1-\frac{v^2}{c^2}},$$ Using Binomial approximation

$$\upsilon = \upsilon_0\left(1-\frac{v^2}{c^2}\right)^{1/2} \simeq \upsilon_0\left(1-\frac{1}{2}\frac{v^2}{c^2}\right)$$

$$\therefore \quad \frac{\upsilon-\upsilon_0}{\upsilon_0} = -\frac{1}{2}\frac{v^2}{c^2}$$

L.H.S gives the fractional change in frequency $\frac{\delta\upsilon}{\upsilon_0}$.

i.e., $$\frac{\delta\upsilon}{\upsilon_0} = -\frac{1}{2}\frac{v^2}{c^2}$$

Multiply numerator an denominator by m, the mass of the atom.

$$\frac{\delta\upsilon}{\upsilon_0} = -\frac{1}{2}\frac{mv^2}{mc^2}$$

According to kinetic theory of gases $\frac{1}{2}mv^2$ is the kinetic energy of atom due to thermal motion. According to statistical mechanics

$$\frac{1}{2}mv^2 = \frac{3}{2}kT$$

where k is the Boltzmann's constant and T is the temperature in kelvin.

$$\therefore \quad \frac{\delta\upsilon}{\upsilon_0} = -\frac{3kT}{2mc^2}$$

substituting the value of $k = 1.38\times10^{-23}\,JK^{-1}$

$m = 1.67\times10^{-27}\text{kg}$ and $c = 3\times10^{8}\text{ms}^{-1}$

we get $\frac{\delta\upsilon}{\upsilon_0} = \frac{-3\times1.38\times10^{-23}\times T}{3\times1.67\times10^{-27}\times9\times10^{16}}$

$$\frac{\delta\upsilon}{\upsilon_0} = -1.377\times10^{-13}T$$

This shows that in order to have an accuracy of 1 part in 10^{13}, it is necessary that the temperature measurement of the radiating atoms should have an accuracy of one kelvin. If we go for an accuracy of 1 part in 10^{15}, T should have an accuracy of 10^{-3} K. It is an herculean task to achieve this.

Relativistic transformation of velocity

Suppose a particle has a velocity u in the rest frame its components along the three coordinates are

$$u_x = \frac{dx}{dt},\ u_y = \frac{dy}{dt} \text{ and } u_z = \frac{dz}{dt}$$

and its velocity in a frame of reference S' moving with a velocity v relative to S along the positive x- direction is u'. Its components are

$$u'_x = \frac{dx'}{dt'},\ u'_y = \frac{dy'}{dt'} \text{ and } u'_z = \frac{dz'}{dt'}$$

We will see that how these component velocities in the two frames are related to each other. From the Lorentz transformation equations we have

$$x' = \frac{x - vt}{\sqrt{1-v^2/c^2}},\ y' = y,\ z' = z,\ t' = \frac{t - vx/c^2}{\sqrt{1-v^2/c^2}}$$

Taking the differentials of above four equations, we have

$$dx' = \frac{dx - vdt}{\sqrt{1-v^2/c^2}} \quad ......(1)$$

$$dy' = dy \quad ......(2)$$

$$dz' = dz \quad ......(3)$$

$$dt' = \frac{dt - vdx/c^2}{\sqrt{1-v^2/c^2}} \quad ......(4)$$

$\dfrac{\text{equation 1}}{\text{equation 4}}$ gives $\quad \dfrac{dx'}{dt'} = \dfrac{dx - v\,dt}{dt - \dfrac{v\,dx}{c^2}} = \dfrac{\left(\dfrac{dx}{dt} - v\right)}{\left(1 - \dfrac{v}{c^2}\dfrac{dx}{dt}\right)}$

$$u'_x = \frac{u_x - v}{1 - \dfrac{u_x v}{c^2}} \qquad \text{...... (5)}$$

$\dfrac{\text{equation 2}}{\text{equation 4}}$ gives $\quad \dfrac{dy'}{dt'} = \dfrac{dy\sqrt{1 - v^2/c^2}}{dt \;- v\,dx/c^2} = \dfrac{\dfrac{dy}{dt}\sqrt{1 - v^2/c^2}}{1 - \dfrac{v\,dx/dt}{c^2}}$

$$u'_y = \frac{u_y\sqrt{1 - v^2/c^2}}{1 - \dfrac{u_x v}{c^2}} \qquad \text{...... (6)}$$

$\dfrac{\text{equation 3}}{\text{equation 4}}$ gives $\quad u'_z = \dfrac{u_z\sqrt{1 - v^2/c^2}}{1 - \dfrac{u_x v}{c^2}} \qquad \text{...... (7)}$

Equations 5, 6 and 7 are called velocity transformation equations.

The inverse velocity trasformation equations are

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2} \qquad \text{...... (8)}$$

$$u_y = \frac{u'_y\sqrt{1 - v^2/c^2}}{1 + \dfrac{u'_x v}{c^2}} \qquad \text{...... (9)}$$

$$u_z = \frac{u'_z\sqrt{1 - v^2/c^2}}{1 + \dfrac{u'_x v}{c^2}} \qquad \text{...... (10)}$$

To see that velocity of light is invariant under the relativistic velocity transformations, let us consider two photons A and B approaching one another as shown in figure.

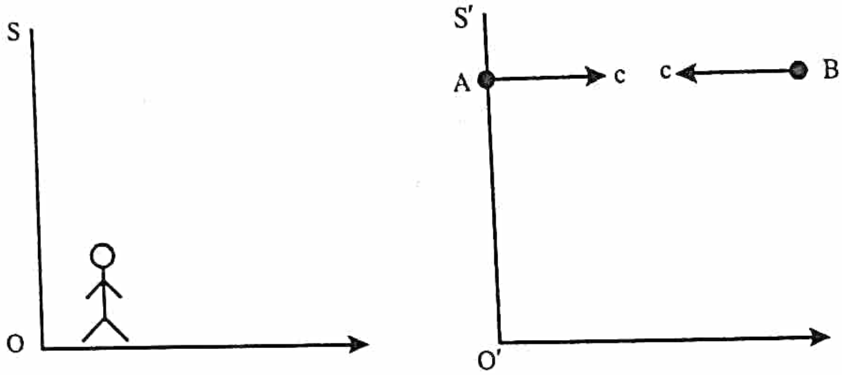


Figure 1.11

We have to calculate the velocity of photon B with respect to A. For this consider an observer in the rest frame S. Let another observer sitting on photon A which is considered as the moving frame S'. i.e. S' frame is moving with speed c. If B is a photon observed by the observer in the rest frame S. Then $u_x = -c$ and $v = c$

$\therefore$ The velocity of the photon B with respect to A

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{-c - c}{1 - \frac{c \times -c}{c^2}} = -c$$

Let us now consider two photons A and B moving in the same direction as shown in figure below.

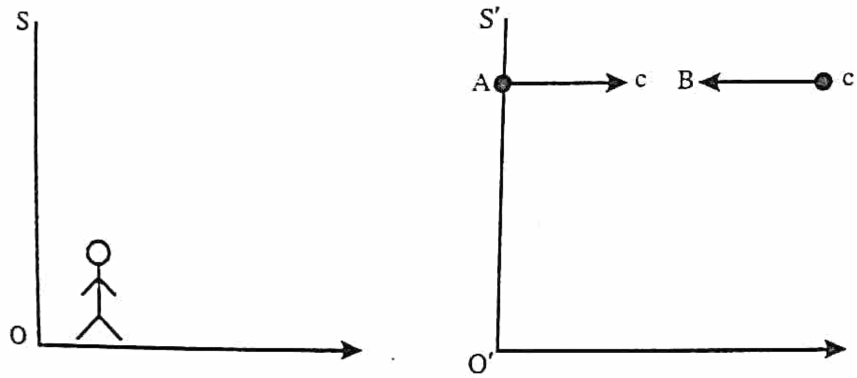


Figure 1.12

Proceeding as described above, we have

$$u_x = c, \quad v = c$$

$$\therefore \quad u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{c - c}{1 - \frac{c \cdot c}{c^2}} = \frac{0}{0}$$

This is indeterminate. To determine the value of u'_x, we take the limit of above equation with $v \to c$.

$$u'_x = Lt_{v \to c} \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = Lt_{v \to c} \frac{c - v}{1 - \frac{v}{c}}$$

Using L'Hospital's rule, we get

$$= \left. \frac{\frac{d}{dv}(c - v)}{\frac{d}{dv}\left(1 - \frac{v}{c}\right)} \right|_{v = c} = \frac{-1}{-\frac{1}{c}} = c$$

This is in well agreement with the second postulate of special theory of relativity.

The relativistic velocity transformation equations were tested experimentally by T. Alvagar at CERN. Alvager used a beam of protons of energy 20 GeV. The protons bombarded a target to produce neutral pions (π°) of energy more than 6 GeV. π° decays into two γ-ray photons. Alvagar determined the velocity of γ-ray photon moving in the forward direction. The velocity of γ-ray photon in a frame at rest with respect to π° (i.e. S' frame) $u'_x = c$. The velocity of π° with respect to $S = 0.99975c$

$\therefore$ Velocity of γ-ray photon with respect to S frame.

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} = \frac{c + 0.99975c}{1 + \frac{0.99975c^2}{c^2}} = c$$

The experimental measurement was carried out by measuring the time taken by the γ-ray to travel between two detectors placed 0.3m apart. The experimental value was in excellent agreement with the velocity calculated using relativistic velocity transformations.

Example 18

Two electrons leave a radio active sample in opposite directions, each having a speed 0.67c with respect to the sample. Calculate the speed of one electron with respect to the other (i) classically and (ii) relativistically.

Solution

Consider one electron as the S-frame, the sample as the S' frame, and the other electron as the object whose speed in the S-frame is to be determined. Then $u' = 0.67c,\ v = 0.67c$.

Using inverse velocity transformation equation we have

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

$$u = \frac{0.67c + 0.67c}{1 + \frac{0.67c \times 0.67c}{c^2}}$$

$$u = \frac{1.34c}{1.45} = 0.92c$$

According to classical relation (Galilean velocity transformation equation)

$$u = u' + v$$

$$u = 0.67c + 0.67c = 1.34c$$

Example 19

Two photons approach each other. What is the velocity of one photon with respect to another.

Solution

Here $u' = c$ and $v = c$

using
$$u = \frac{u' + v}{1 + \frac{uv}{c^2}} = \frac{c + c}{1 + \frac{c^2}{c^2}} = \frac{2c}{2} = c$$

Example 20

If a photon traverses the path in such a way that it moves in $x' - y'$ plane and makes an angle θ with the axis of the frame S', then prove for frame S,

$$u_x^2 + u_y^2 = c^2$$

Solution

and $$\left.\begin{aligned} u'_x &= c\cos\theta \\ u'_y &= c\sin\theta \end{aligned}\right] \text{ given}$$

using $$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}}$$

and $$u_y = \frac{u'_y\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{u'_x v}{c^2}}$$

We get $$u_x = \frac{c\cos\theta + v}{1 + \frac{v}{c}\cos\theta} \qquad \text{...... (1)}$$

$$u_y = \frac{c\sin\theta\sqrt{1\frac{v^2}{c^2}}}{1 + \frac{v}{c}\cos\theta} \qquad \text{...... (2)}$$

Squaring and adding eqs (1) and (2) we get

$$u_x^2 + u_y^2 = \frac{(c\cos\theta + v)^2}{\left(1 + \frac{v}{c}\cos\theta\right)^2} + \frac{c^2\sin^2\theta\left(1 - \frac{v^2}{c^2}\right)}{\left(1 + \frac{v}{c}\cos\theta\right)^2}$$

$$u_x^2 + u_y^2 = \frac{c^2\cos^2\theta + v^2 + 2vc\cos\theta + c^2\sin^2\theta - v^2\sin^2\theta}{\left(1 + \frac{v}{c}\cos\theta\right)^2}$$

$$u_x^2 + u_y^2 = \frac{c^2 + v^2 + 2v\cos\theta - v^2\sin^2\theta}{\left(1 + \frac{v}{c}\cos\theta\right)^2}$$

$$u_x^2 + u_y^2 = \frac{c^2 + v^2 + 2vc\cos\theta - v^2(1-\cos^2\theta)}{\left(1+\frac{v}{c}\cos\theta\right)^2}$$

$$u_x^2 + u_y^2 = \frac{c^2 + 2vc\cos\theta + v^2\cos^2\theta}{\left(1+\frac{v}{c}\cos\theta\right)^2}$$

$$u_x^2 + u_y^2 = \frac{c^2\left(1+\frac{2v}{c}\cos\theta + \frac{v^2}{c^2}\cos^2\theta\right)}{\left(1+\frac{v}{c}\cos\theta\right)^2}$$

$$u_x^2 + u_y^2 = \frac{c^2\left(1+\frac{v}{c}\cos\theta\right)^2}{\left(1+\frac{v}{c}\cos\theta\right)^2} = c^2$$

Thus $u_x^2 + u_y^2 = c^2$

Example 21

Two velocities 0.8c each are inclined to one another at angle of $30°$. Obtain the value of the result.

Solution

Using $$u_x = \frac{u_x' + v}{1+\frac{u_x' v}{c^2}}$$

$u_x' = 0.8c \times \cos 30$ and $v = 0.8c$

We get $$u_x = \frac{0.8c\cos 30 + 0.8c}{1+\frac{0.8c\cos 30 \times 0.8c}{c^2}} = 0.96c$$

and $$u_y = \frac{u'_y\sqrt{1-\frac{v^2}{c^2}}}{1+\frac{u'_x v}{c^2}} = \frac{0.8c\sin 30\sqrt{1-0.8^2}}{1+\frac{0.8c\cos 30\times 0.8c}{c^2}}$$

$$= 0.15c$$

∴ $$u = \sqrt{u_x^2+u_y^2} = \sqrt{(0.96c)^2+(0.15c)^2} = 0.97c$$

Example 22

Two rods having the same length l_0 move lengthwise towards each other parallel to a common axis with the same velocity v relative to lab frame. What is the length of each rod in the frame fixed to the other rod.

Solution

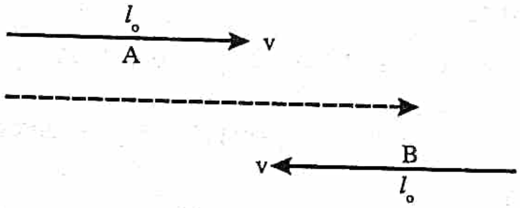


Figure 1.13

Velocity of A with respect t_0 B,

$$v_{AB} = \frac{v_A + v_B}{1+\frac{v_A v_B}{c^2}} = \frac{v+v}{1+\frac{v^2}{c^2}}$$

$$v_{AB} = \frac{2v}{1+\frac{v^2}{c^2}}$$

The length of the rod as measured by B is

Using $l = l_0\sqrt{1-\frac{v_{AB}^2}{c^2}}$

$$l = l_0\sqrt{1-\left(\frac{2v}{1+\frac{v^2}{c^2}}\right)^2 \cdot \frac{1}{c^2}}$$

$$l = l_0\sqrt{1-\frac{4v^2.c^2}{(c^2+v^2)^2}}$$

$$l = l_0\sqrt{\frac{(c^2+v^2)^2-4v^2c^2}{(c^2+v^2)^2}}$$

$$\therefore \quad l = l_0\frac{\left(1-\frac{v^2}{c^2}\right)}{1+\frac{v^2}{c^2}}$$

Speed of light in a medium

Consider a tube filled with water at rest. The velocity of light in the water with respect to laboratory frame is $\frac{c}{n}$, where n is the refractive index of water. Recall that

$$n = \frac{\text{velocity of light in vacuum}}{\text{velocity of light in medium}}$$

Suppose the water is flowing through the tube with speed v. Then what is the speed of light in moving water with respect to lab frame?

Using velocity addition formula, we have

$$u = \frac{u'+v}{1+\frac{u'v}{c^2}}$$, where , u' is the velocity of light with respect to water

$$\therefore \quad u = \frac{\frac{c}{n}+v}{1+\frac{v}{nc}}$$

$$u = \frac{c}{n}\frac{\left(1+\frac{nv}{c}\right)}{\left(1+\frac{v}{nc}\right)}$$

$$u=\frac{c}{n}\left(1+n\frac{v}{c}\right)\left(1+\frac{v}{nc}\right)^{-1}$$

using Binomial approximation to the last term, we get

$$u=\frac{c}{n}\left(1+\frac{nv}{c}\right)\left(1-\frac{v}{nc}\right)$$

$$u=\frac{c}{n}\left(1+\frac{nv}{c}-\frac{v}{nc}-\frac{v^2}{c^2}\right)$$

Neglecting $\frac{v^2}{c^2}$, we get

$$u=\frac{c}{n}\left(1+\frac{nv}{c}-\frac{v}{nc}\right)$$

or $$u=\frac{c}{n}+v-\frac{v}{n^2}$$

or $$u=\frac{c}{n}+\left(1-\frac{1}{n^2}\right)v$$

This shows that the velocity of light in moving water with respect to lab frame is increased. In other words light appears to be dragged by the water by a factor $1-\frac{1}{n^2}$ times the speed of water. This effect is entirely due to relativity.

The Doppler effect

C.J. Doppler in 1842 observed that **whenever there is a relative motion between the source of sound and observer there is an apparent changes in frequency of the source of sound heard by the observer. This phenomenon is called Doppler effect.**

Doppler's effect is easily observed by a person standing on a platform when an engine sounding horn passes by. It is observed that the pitch(frequency) of engine appears to increase when the train approaches the person and appears to decrease when the train receeds away from the person. Though Doppler's effect is most commonly and easily observed with sound waves, all types of waves including light waves exhibit Doppler effect. Here we discuss Doppler effect in sound as well as in light waves.

Doppler shift in sound

Consider a source of sound S and an observer O. Let v and w be the velocities of the source and sound respectively. υ be the frequency of sound from the source.

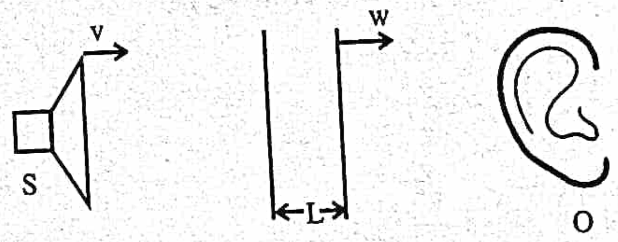


Figure 1.14

In time t the sound travels a distance of wt, consider the sound as a series of pulses separated by a time $\tau = \frac{1}{\upsilon}$.

If L be the separation of two pulses, then the number of pulses reaching the observer

$$= \frac{\text{Distance travelled}}{\text{pulse separation}}$$

$$= \frac{wt}{L}$$

$\therefore$ The number of pulses received per second $= \frac{w}{L}$

The number of pulses received per second by the observer is the frequency heard by the observer. it is denoted by υ'

i.e, $\upsilon' = \frac{w}{L}$

Now we can calculate L.

To determine L, consider a pulse emitted at $t = 0$ and the next pulse emitted at $t = \tau$. During this time τ first pulse travelled a distance of $w\tau$ and the source travelled a distance of $v\tau$.

$\therefore$ Then distance between the two pulses,

$$L = w\tau - v\tau = (w - v)\tau$$

$$L = \frac{(w - v)}{\upsilon} \left(\because \ \upsilon = \frac{1}{\tau}\right)$$

$\therefore$
$$\upsilon' = \frac{w}{L} = \frac{w}{\frac{(w-v)}{\upsilon}} = \upsilon \frac{w}{w - v}$$

or
$$\upsilon' = \upsilon \frac{1}{1 - \frac{v}{w}}$$

This is the expression for the apparent frequency heard by the observer when source is moving towards the observer. Obviously $\upsilon' > \upsilon$

When the source is moving away from the observer replace v by $-v$. Thus

$$\upsilon' = \upsilon \frac{1}{1 + \frac{v}{w}} \text{ here } \upsilon' < \upsilon.$$

Now we consider a different situation where the source of sound is at rest and the observer is moving towards the source with a speed v.

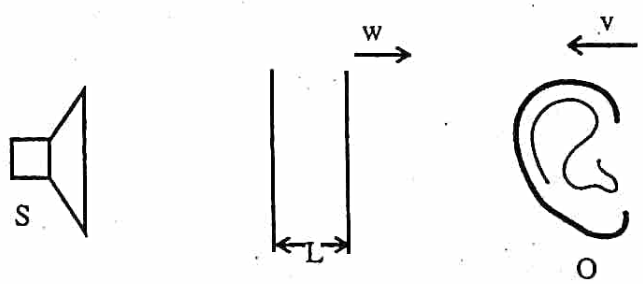


Figure 1.15

Since the source is at rest the distance between the two pulses, $L = w\tau = \frac{w}{\upsilon}$.

In time t, the number of pulses received by the observer $= \frac{(w+v)t}{L}$

$\therefore$ The number of pulses received by the observer in one second $= \frac{(w+v)}{L}$

This is nothing but the frequency heard by the observer (υ').

i.e. $$\upsilon' = \frac{(w+v)}{L}$$

Substituting for L, we get $\upsilon' = \frac{(w+v)}{w/\upsilon} = \upsilon\frac{(w+v)}{w}$

$$\upsilon' = \upsilon\left(1+\frac{v}{w}\right)$$

This is the expression for apparent frequency heard by the observer, when observer is moving towards the source.

When the observer is moving away from the source replace v by $-v$.

Thus $$\upsilon' = \upsilon\left(1-\frac{v}{w}\right)$$

The Doppler effect in sound gives us an important information. That is, knowing υ, v and w we can calculate υ' from which we can tell whether it is the observer or source is moving.

If this result is applicable to light waves (relativistic Doppler effect) then we would be able to distinguish between inertial frames which one is at rest or motion. This is contrary to the principle of relativity that it is not possible to distinguish between two inertial frames at rest and in uniform motion. To resolve this we go for relativistic Doppler effect.

Relativistic Doppler effect

Consider a light source emitting pulses of frequency υ in its rest frame $\left(\upsilon = \frac{1}{\tau_0}\right)$. The source is moving towards an observer with velocity.

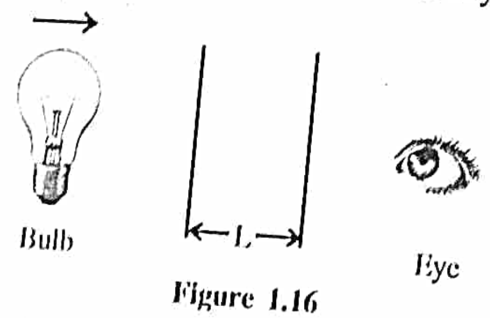


Figure 1.16

It is due to time dilation, the period in the observers rest frame is

$$\tau = \gamma\, \tau_0$$

Since the speed of light is a universal constant, the pulse arrive at the observer with velocity c

The frequency of the pulses observed by the observer is

$\upsilon' = \frac{c}{L}$, where L is the separation between the two pulses with respect to the observer.

Since the source is moving towards the observer

$$L = c\tau - v\tau \;= (c-v)\tau$$

$$\therefore \quad \upsilon' = \frac{c}{(c-v)\tau} = \frac{c}{(c-v)}\frac{1}{\gamma\tau_0}$$

$$\upsilon' = \upsilon\sqrt{1-v^2/c^2}\,.\,\frac{c}{c-v}$$

or

$$\upsilon' = \upsilon\frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}}$$

$$\upsilon' = \upsilon\sqrt{\frac{1+v/c}{1-v/c}}$$

This is the expression for the observed frequency. Obviously $\upsilon' > \upsilon$ as we expect since the source is moving towards the observer.

When the source is moving away from the observer replace v by $-v$.

i.e.,

$$\upsilon' = \upsilon\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \quad \text{here } \upsilon' < \upsilon$$

Note: It may be noted that the relativistic Doppler frequency is the geometric mean of two classical results.

Doppler effect for an observer off the line of motion

So far we considered the Doppler effect for a source and observer along the line of motion. It may not be always so. So we consider the general case.

Consider an observer is at angle θ from the line of motion. In this case the only parameter that get affected is the distance between the two pulses L.

When the source and observer are along the line of motion, we have

$$L = c\tau - v\tau$$

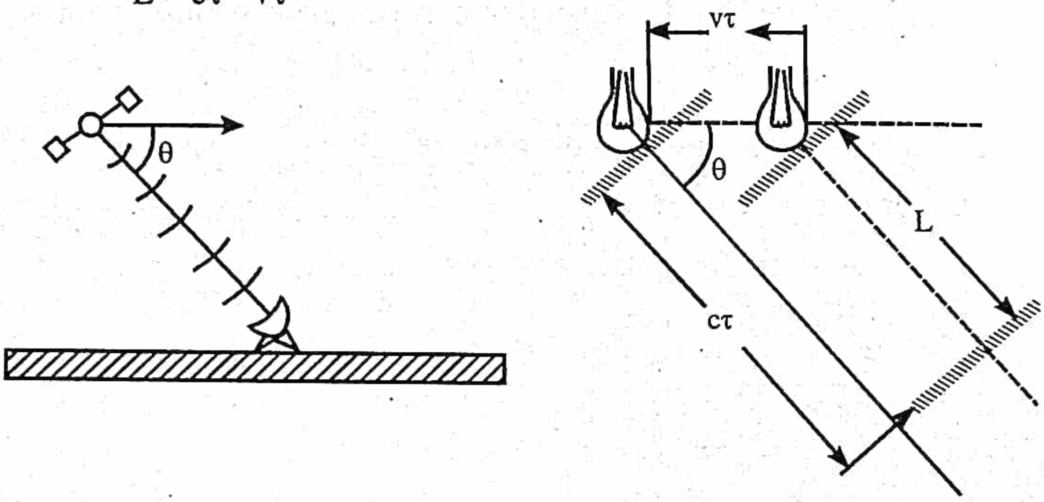


Figure 1.17

when the observer is at angle θ from the line of motion, the distance travelled by light is $c\tau$ as before since light is pervading every where. But the distance travelled by the source towards the observer is only $v\cos\theta\ \tau$. Thus

$$L = c\tau - v\cos\theta\ \tau = (c - v\cos\theta)\tau$$

Using
$$\upsilon' = \frac{c}{L} = \frac{c}{(c - v\cos\theta)\gamma\tau_0}$$

$$\upsilon' = \upsilon\frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}\cos\theta}$$

It may be noted that θ is the angle measured in the rest frame of the observer.

When $\theta = 0$

$$\upsilon' = \upsilon\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

We get back our old result. This is called the longitudinal Doppler effect.

$$\theta = 90°, \quad \upsilon' = \upsilon\sqrt{1 - \frac{v^2}{c^2}}$$

This is called transverse Doppler effect, a phenomenon not found in Doppler effect of sound. It is solely due to time dilation. Doppler effect of light has been experimentally confirmed by Ives and Stilwell in the year 1938.

Note: Comparing Doppler effect of sound and light we can see that both formulae are the same to first order approximation in $\frac{v}{c}$. So in order to differentiate between them the effects of order $\frac{v^2}{c^2}$ has to be considered, which is a difficult task.

Application of Doppler effect – Doppler navigation

Doppler effect can be used to track a moving body such as a satellite or a fighter plane from a reference point on the earth. The precision of this tracking is fantastically high. For example the position of a satellite 10^8 m away can be determined to a fraction of a centimeter. This method was effectively used in II world war to locate enemy fighter planes.

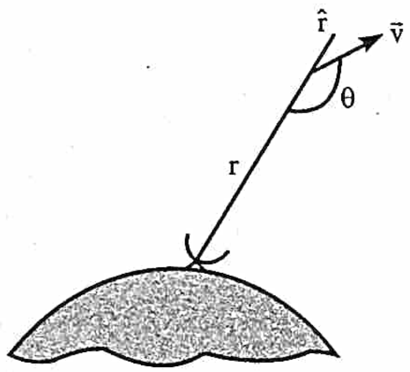


Figure 1.18

Consider a satellite moving with a velocity v at angle θ with the line of sight with respect to the ground observation point. An oscillator on the satellite broadcasts a signal with a frequency υ with respect to the satellite.

According to Doppler formula the frequency υ' received by the ground station is

$$\upsilon' = \upsilon \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v}{c}\cos\theta}$$

since $v \ll c$, $\frac{v^2}{c^2}$ can be neglected.

$$\therefore \quad \upsilon' = \frac{\upsilon}{1-\frac{v}{c}\cos\theta} = \upsilon\left(1-\frac{v}{c}\cos\theta\right)^{-1}$$

or
$$\upsilon' = \upsilon(1+\frac{v}{c}\cos\theta)$$

or
$$\upsilon' - \upsilon = \upsilon\frac{v}{c}\cos\theta$$

This shift in frequency (Doppler shift) can be measured from the ground station. For this we keep an oscillator identical to the one in the satellite, and by simple electronic methods the difference in frequency $\upsilon' - \upsilon$ (beat frequency) can be measured.

Now we have to calculate distance of the satellite by knowing $\upsilon' - \upsilon$.

The velocity of the satellite is $\bar{v} = v_r\ \hat{r} + v_\theta\ \hat{\theta}$, when $v_r = \frac{dr}{dt}$ and $v_\theta = r\frac{d\theta}{dt}$.

So
$$\hat{r}\cdot\bar{v} = \frac{dr}{dt}\hat{r}\cdot\hat{r}$$

or
$$\frac{dr}{dt} = \hat{r}\cdot\bar{v}$$

$$\frac{dr}{dt} = v\cos(180-\theta) = -v\cos\theta$$

But
$$v\cos\theta = \frac{c}{\upsilon}(\upsilon' - \upsilon)$$

$$\therefore \quad \frac{dr}{dt} = -\frac{c}{\upsilon}(\upsilon' - \upsilon)$$

$$\frac{dr}{dt} = -\lambda\ (\upsilon' - \upsilon)$$

Where λ is the wave length of the pulse emitted from the oscillator.

Integrating the above equation with respect to time within limits t_a and t_b, we get

$$r_b - r_a = -\lambda \int_{t_a}^{t_b} (\upsilon' - \upsilon)dt$$

The integral on the R.H.S is the number of cycles of beat frequency which occurs in the interval $t_b - t_a$.

i.e, $\quad r_b - r_a = \lambda N_{ba}$

Knowing λ and the number of beats received we calculate $r_b - r_a$.

Satellite communication systems operate at a typical wavelength 10cm and since the beat signal can be measured to a fraction of a cycle, satellites can be tracked to about 1cm.

If the satellite and ground station oscillators do not match, a two way Doppler tracking system can be used in which a signal from the ground is broadcast to the satellite which then amplifies it and relays back to the ground station. This will double the Doppler effect there by increasing the resolution by a factor of two.

Example 23

One of the most prominent spectral lines of hydrogen is H_α line, a bright red line with a wavelength of 656.1×10^{-9}m. What is the expected wavelength of the H_α line from a star receding with a speed of $3000 kms^{-1}$.

Solution

$$\lambda = 656.1\times10^{-9} m, \ v = 3\times10^{6} ms^{-1}$$

When the source is receding away, we have

$$\upsilon' = \upsilon \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$$

using $\quad \upsilon' = \dfrac{c}{\lambda'}$ and $\upsilon = \dfrac{c}{\lambda}$

$$\frac{c}{\lambda'} = \frac{c}{\lambda}\sqrt{\frac{1-v/c}{1+v/c}}$$

or $$\lambda' = \lambda\sqrt{\frac{1-v/c}{1+v/c}}$$

$$\lambda' = 656.1\times10^{-9}\sqrt{\frac{1+\frac{3\times10^{6}}{3\times10^{8}}}{1-\frac{3\times10^{6}}{3\times10^{8}}}}$$

$$\lambda' = 656.1\times10^{-9}\sqrt{\frac{1+10^{-2}}{1-10^{-2}}}$$

$$\lambda' = 656.1\times10^{-9}\sqrt{\frac{1.01}{0.99}}$$

$$\lambda' = 656.1\times10^{-9}\times1.01$$

$$\lambda' = 662.7\times10^{-9}\,\text{m}$$

Example 24

The H_α line measured on earth from opposite ends of equator differ in wave length by $9\times10^{-12}\,\text{m}$. Assume that the effective is caused by rotation of the Sun, find the period of rotation.

The radius of the Sun is $696.34\times10^{6}\,\text{km}$ $\lambda = 656.1\times10^{-9}\,\text{m}$

Solution

Let λ_1' be the observed wave length of H_α line coming from A towards the observer

i.e., $$\lambda_1' = \lambda\sqrt{\frac{1-v/c}{1+v/c}} \simeq \lambda\sqrt{\left(1-\frac{v}{c}\right)\left(1+\frac{v}{c}\right)^{-1}}$$

$$\lambda_1' \simeq \lambda\sqrt{1-\frac{2v}{c}} = \lambda\left(1-\frac{v}{c}\right).$$

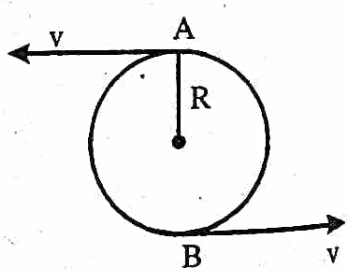


Figure 1.19

Similarly λ_2' be the observed wave length of the line moving away from B, we get

$$\lambda_2' \approx \lambda\left(1+\frac{v}{c}\right)$$

$$\therefore \quad \lambda_2' - \lambda_1' = \lambda\left(1+\frac{v}{c}\right) - \lambda\left(1-\frac{v}{c}\right)$$

$$\lambda_2' - \lambda_1' = 2\lambda\frac{v}{c}$$

$$v = \frac{c(\lambda_2' - \lambda_1')}{2\lambda}$$

$$v = \frac{3\times10^8\times9\times10^{-12}}{2\times656.1\times10^{-9}}$$

$$v = 2.058\text{x}10^3\,\text{ms}^{-1}.$$

Using $v = R\omega$

$$\omega = \frac{v}{R} = \frac{2.058\times10^3}{696.340\times10^6}$$

$$\omega = 2.955\times10^{-6}\ \text{radians}$$

$$\therefore \quad T = \frac{2\pi}{\omega} = \frac{2\times3.14}{2.955\times10^{-6}}$$

$$T = 2.125\times10^6\,\text{s}$$

$$T = \frac{2.125\times10^6}{60\times60\times24}\ \text{days}$$

$$T = 24\cdot59\ \text{days}$$

Example 25

A rocket ship is receding from the earth at a speed of 0.2c. A light in the rocket ship appears blue to passengers on the ship what colour would it appear to be to an observer on the earth. $\lambda_{\text{blue}} = 470\text{nm}$

Solution

$$v = 0.2c,\ \lambda_{\text{blue}} = 470\text{nm}$$

Using $\lambda' = \lambda\sqrt{\dfrac{1+\frac{v}{c}}{1-\frac{v}{c}}}$, $\dfrac{v}{c} = 0.2$

$$\lambda' = 470\times10^{-9}\sqrt{\frac{1+0.2}{1-0.2}}$$

$$\lambda' = 470\times10^{-9}\times1.225$$

$\lambda' = 575.6\text{mm}$ The colour appears to be yellow.

Twin Paradox

Consider two synchronised clocks. One is kept on earth the other is taken in a fast moving space ship. If the fast moving clock is brought back to earth after a long time we can see that the time elapsed in moving clock will be less than the time elapsed in clock on earth. This is because a moving clock runs slow. Actually there is no difference between a physical clock and a biological clock accordingly heart beats of a person can be taken as clock. i.e. when a person moves with high speed, his heart beat will be slower. If his age is counted with reference to his heart beat his age will be growing slower. Here comes the twin paradox.

Consider two identical twins A and B, if A goes about at a high speed v in a rocket and B stays behind on earth. When A returns to earth he will be younger than B. In relativity situations are interchangeable. Hence we can assume that A is rest and B is moving with velocity -v. After return B will be found to be younger than A. That is each should find the other younger. A logical contradiction. i.e. a paradoxical statement. This is called twin paradox. This paradox comes because we assumed that twins situations are symmetrical and interchangeable, an assumption that is not correct. Thus there is no paradox at all.

Quantitative analysis of ageing of twins

The twin B stays on earth observes that A travels away a distance L in time $t = \dfrac{L}{v}$, where t is the taken by the twin A to travel forward with respect to B. After t, twin A rapidly reverses his motion and returns with the same velocity. The time for return trip is also t. If we neglect the time for turn around, the total time taken by A with respect to B is $t_B = 2t$

It is due to time dilation A's clock runs slow as far as B is concerned. The time measured by A for the round trip is

$$t'_A = \frac{t_B}{\gamma} = t_B\sqrt{1-\frac{v^2}{c^2}}$$

$$\therefore \qquad \frac{\text{Age of A}}{\text{Age of B}} = \frac{t'_A}{t_B} = \sqrt{1-v^2/c^2}$$

From this B concludes that A is younger.

Now we calculate the age of B as far as A is concerned. Twin A sees that B going away for distance L with velocity –Vand return. This takes time $t_A = 2t$ on 's clock and A sees time t'_B elapse on B's clock, then

$$t'_B = \frac{t_A}{\gamma} = t_A\sqrt{1-\frac{v^2}{c^2}}$$

$$\therefore \qquad \frac{\text{Age of A}}{\text{Age of B}} = \frac{t_A}{t'_B} = \frac{1}{\sqrt{1-v^2/c^2}}$$

From this A concludes that B is younger. This is the paradox. A thinks that B is younger and B thinks that A is younger.

Example 26

A young man voyages to the nearest star, α centauri, 4.3 light years away. He travels in space ship at a velocity of $\frac{c}{5}$. When he returns to earth, how much younger is he than his twin brother who stayed at home.

Solution

The time taken for the round trip $t = \frac{2\times 4.3\times c}{v}$ years.

$$= \frac{2\times 4.3\times c}{\frac{c}{5}} = 43 \text{ years}$$

Using time dilation $\quad t' = t\sqrt{1-\frac{v^2}{c^2}} \quad = 43\sqrt{1-\frac{1}{25}}$

$$t' = 43\sqrt{\frac{24}{25}} = 43 \times .97979$$

$$t' = 42.13 \text{ years}$$

$$\therefore \quad \text{Age difference} = t - t'$$

$$= 43 - 42.131$$

$$= 0.87 \text{ years}$$

$$= 0.87 \times 12 \text{ months}$$

$$\simeq 10 \text{ months}$$

i.e., the voyager is 10 months younger.

Relativistic momentum

Here we develop the dynamics of special theory of relativity. There are several ways to do this. One approach is to develop formal procedure for writing the laws of physics in a form which satisfies the postulates of relativity. We start with conservation of momentum, because this is one of the basic laws of nature. We seek what modifications are required to preserve this principle in relativity. This will tell us we must modify our idea of mass to preserve conservation of momentum in relativity. Once this is achieved we can extend this to develop the entire dynamics in relativity.

Concept of mass and momentum in relativity

For this consider a glancing (oblique) elastic collision of two identical particles A and B each having mass m and move with velocity u in opposite directions as shown in figure.

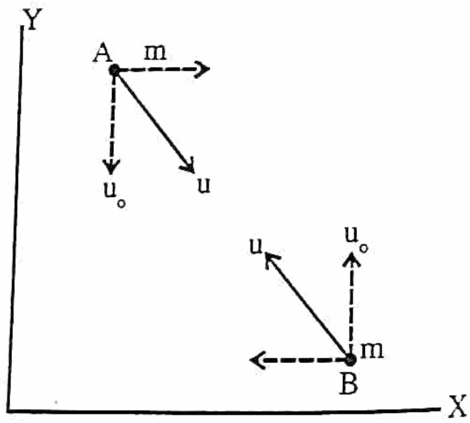


Figure 1.20

After some time particles collide and move in opposite directions. Here we are going to view the collision process (before and after) with two special frames A and B. The A-frame moves with a velocity same as that of the x-component velocity of the particle A. Thus with respect to A frame particle A has only the y-component velocity of the particle A. Let it be u_0. The B-frame moves with a velocity same as that of the x-component velocity of the particle B. With respect A-frame, the particle B has x component velocity and y component velocity.

The x component velocity of the particle B with respect to $A = v$

v is actually the relative velocity between the two frames.

From the velocity addition theorem, we can calculate the velocity of the particle B with respect to the A frame.

Using $$u'_y = \frac{u_y\sqrt{1-v^2/c^2}}{1-\frac{u_x v}{c^2}}$$

Here $u_y = u_0$, $v = v$ and $u_x = 0$

Thus $$u'_y = \frac{u_y}{\gamma}, \quad v = \frac{1}{\sqrt{1-v^2/c^2}}$$

Since the particles are identical, their velocities are the same and the situations are symmetric, we can write down the velocities of the particles A and B with respect to B-frame. The x-component velocity of the particle B with respect to frame $A = u_0$

The x-component velocity of the particle A with respect to B-frame = v

The y-component velocity of the particle A with respect to B-frame $\frac{u_0}{\gamma}$

The velocities of particles A and B before collision with respect to A-frame and B-frame are shown in figure.

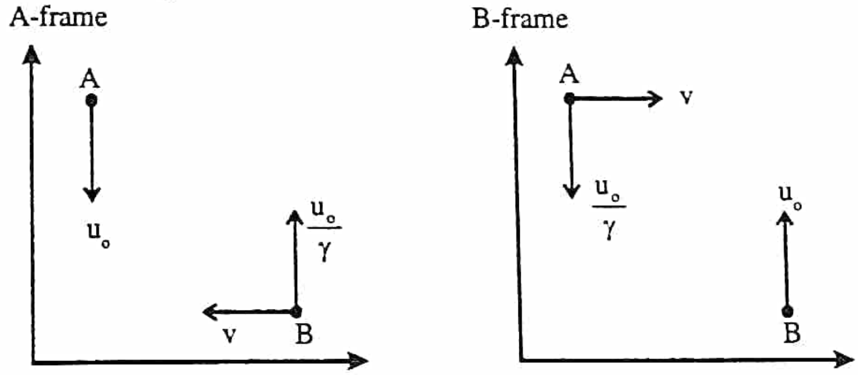


Figure 1.21: Before collision

After collision the x-component velocities remain as such and y-component velocities reversed their directions. If the y-speeds of particles in their own frame is u'

then the y-speed of other particle is $\frac{u'}{\gamma}$. The situation after collision is depicted in figure below.

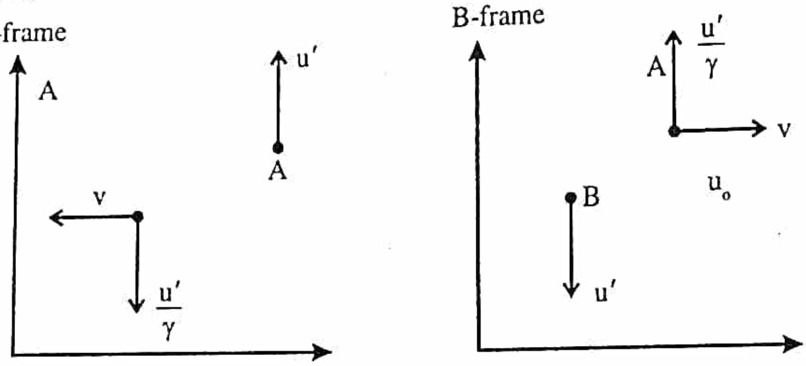


Figure 1.22: After collision

Our aim is to find a conserved quantity analogous to classical momentum. We define the momentum as the product of mass (which is a function of particles velocity w)and velocity w.

i.e $\quad \bar{p} = m(w)\bar{w}$

where m(w) is a scalar quantity to be determined.

In A-frame the velocity of the particle before collision along x-direction is entirely due to particle B. The speed of particle B with respect to A - frame is $w = \left(v^2 + \frac{u_0^2}{\gamma^2}\right)^{\frac{1}{2}}$ before collision. After collision A is $w' = \left(v^2 + \frac{u'^2}{\gamma^2}\right)^{\frac{1}{2}}$. That is before collision m is a function of w and after collision it is a function of w' along x-direction. Applying law of conservation of momentum along x-direction in the A-frame.

Momentum before collision in the x direction in the frame A

= momentum after collision in the x-direction in the frame A

i.e., $\quad m(w)v = m(w')v$

Thus we get $w = w'$

Applying law of conservation of momentum along the y-direction in the A-frame. i.e., Momentum before collision in the y-direction in the A-frame = Momentum after collision in the y-direction in A-frame.

i.e., $-m(u_0)+m(w)\frac{u_0}{\gamma}=m(u')u'-m(w')\frac{u'}{\gamma}$

we proved that $\quad w=w'$

$$\left(v^2+\frac{u_0^2}{\gamma^2}\right)^{\frac{1}{2}}=\left(v^2+\frac{u'^2}{\gamma^2}\right)^{\frac{1}{2}}$$

This implies that $u^0=u'$

Re-writing the above equation, we get

$$-m(u_0)u_0+m(w)\frac{u_0}{\gamma}=m(u_0)u_0-m(w)\frac{u_0}{\gamma}$$

or $\quad 2m(w)\frac{u_0}{\gamma}=2m(u_0)u_0$

or $\quad m(w)=vm(u_0)$

In the limit, when $u_0\to 0$, $m(u_0)\to m(0)$ we call it as the rest mass of the particle denoted by m_0.

We have $w=\left(v^2+\frac{u_0^2}{\gamma^2}\right)^{\frac{1}{2}}$

when $u_0\to 0$, $\quad w=v$

∴ we have $m(v)=\gamma m_0$

or $$m(v)=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

It shows that in relativity mass depends on velocity.

Now we are in a position to get an expression for momentum.

$$\bar{p}=m(u)\bar{u}$$

$$p=\frac{m_0\ \bar{u}}{\sqrt{1-u^2/c^2}}=\gamma m_0 u=m\bar{u}.$$

Velocity dependence of electrons mass

The effect of velocity on the electrons mass was experimentally detected by Bucherer and he verified the relativistic mass formula. For this he designed an experimental setup as shown in figure below.

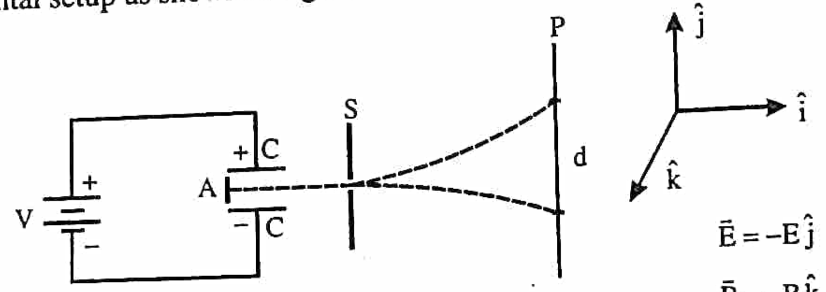


Figure 1.23

The whole apparatus is arranged in an evacuated chamber.

It consists of a source of electron A (radium salt) which emits β-rays which have broad energy spectrum of the order of 1 MeV. These electrons are passed through a velocity filter which can select a monoenergetic electrons. Velocity selector consists of two parallel metal plates C connected to a battery V. i.e, electrons are subjected to an electric field $\vec{E}(-E\,\hat{j})$. A transverse magnetic field $\vec{B}(-B\,\hat{k})$ is also applied to the electrons. Now E, B and v (velocity of electrons) are mutually perpendicular. Now the electric experience electric force qE and magnetic force $q(v \times B)$.

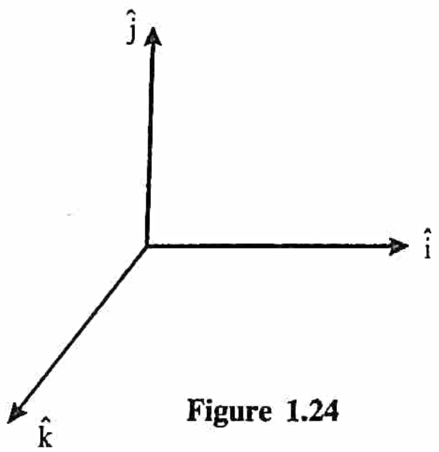


Figure 1.24

The electric force on the charge is $\vec{F}_e = qE\hat{j}$

The magnetic force on the charge

$$\vec{F}_m = qv\hat{i} \times -B\hat{k}$$

$$\vec{F}_m = +qvB\hat{j}$$

Adjust E and B such that the total force on the charge is zero. We have

$$\vec{F}_T = \vec{F}_e + \vec{F}_m$$

When $\vec{F}_T = 0$, we get

$$-\vec{F}_e = \vec{F}_m$$

or $$--qE\hat{j} = qV\hat{j}$$

or $$v = \frac{E}{B}.$$

When the electrons experience no force they pass through the slit S undeflected all having the same velocity $v = \frac{E}{B}$. This is the principle of velocity selector.

Beyond S only perpendicular magnetic field acts. When a charged particle q, moves in a perpendicular magnetic field B, it moves along a circular path. The centripetal force $\frac{mv^2}{r}$ required is supplied by magnetic Lorentz force qvB.

i.e. $$\frac{mv^2}{r} = qvB$$

or $$r = \frac{mv}{qB}$$

using $$v = \frac{E}{B}$$

Radius of curvature $r = \frac{mE}{qB^2}$

Finally the electrons are allowed to fall on a photograph plate P. By reversing $\vec{E}$ and $\vec{B}$ the sense of deflection is reversed. From the total deflection d, and using the geometry of the apparatus, we can evaluate r. By finding r for different velocities, the velocity dependence of $\frac{m}{q}$ can be studied.

From his observations he found that $\frac{q}{m}$ is not constant. $\frac{q}{m}$ was found to vary with velocity. Since charge is independent of velocity, the variation of $\frac{v}{m}$ can be attributed to variation in m alone, Bucherer replaced m by $\frac{m_0}{\sqrt{1-v^2/c^2}}$ and drawn a graph between $\frac{m}{m_0}$ and $\frac{v}{c}$ using his experimental data. On the same graph he plotted $\frac{m}{m_0}$ versus $\frac{v}{c}$ using $m = m_0\sqrt{1-v^2/c^2}$, surprisingly the graphs were one and the same.

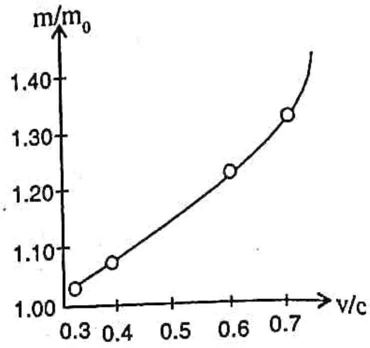


Figure 1.25

The variation of mass with velocity is shown in figure 1.25.

Example 27

Find the velocity at which the mass of a particle is double its rest mass.

Solution

$$m = 2m_0 \text{ (given)}$$

using
$$m = \frac{m_0}{\sqrt{1-v^2/c^2}}$$

$$2m_0 = \frac{m_0}{\sqrt{1-v^2/c^2}}$$

$$2 = \frac{1}{\sqrt{1-v^2/c^2}} \quad \text{squaring on both sides}$$

$$4 = \frac{1}{1-v^2/c^2}$$

$$1-\frac{v^2}{c^2} = \frac{1}{4}$$

$$1-\frac{1}{4} = \frac{v^2}{c^2}$$

$$\frac{3}{4} = \frac{v^2}{c^2}$$

$$\frac{\sqrt{3}}{2}=\frac{v}{c}$$

$$v=\frac{\sqrt{3}}{2}c=0\cdot 866c=0\cdot 866\times 3\times 10^8$$

$$=2\cdot 598\times 10^8 ms^{-1}$$

Example 28

A man weighing 60 kg on the ground. When he is in a space ship in motion his mass is 65 kg measured by an observer on the ground. What is the speed of the space ship.

Solution

$$m_0=60 kg$$

$$m=65 kg$$

Using $$m=\frac{m_0}{\sqrt{1-v^2/c^2}}$$

$$\sqrt{1-\frac{v^2}{c^2}}=\frac{60}{65}=\frac{12}{13} \text{ squaring}$$

$$1-\frac{v^2}{c^2}=\frac{144}{169}$$

$$1-\frac{144}{169}=\frac{v^2}{c^2}$$

$$\frac{25}{169}=\frac{v^2}{c^2}$$

$$\frac{5}{13}=\frac{v}{c}$$

$$v=\frac{5}{13}c=\frac{5}{13}\times 3\times 10^8=\frac{15}{13}\times 10^8$$

$$v=1\cdot 154\times 10^8 ms^{-1}.$$

Mass - Energy relation

The kinetic energy K of a particle is defined as the work done by an external force in increasing the speed of the particle from zero to some value v.

i.e., $$K = \int_0^v \vec{F} \cdot d\vec{x}$$

But $$F = \frac{dp}{dt} = \frac{d(mv)}{dt}$$

$$K = \int_0^v \frac{d}{dt}(mv) \cdot dx = \int_0^v d(mv) \cdot \frac{dx}{dt} = \int_0^v d(mv) \cdot v$$

$$K = \int_0^v (m\,dv + v\,dm) \cdot v = \int_0^v (mv\,dv + v^2 dm) \qquad \text{...... (1)}$$

(Here m and v are variables)

Using $m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$ squaring and rearranging, we have

$$m^2 (c^2 - v^2) = m_0^2 c^2$$

Taking the differenitials on both sides, we get

$$2m\,dm\,c^2 - 2mv^2 dm - 2m^2 v\,dv = 0$$

$$c^2 dm - v^2 dm - mv\,dv = 0$$

$$mv\,dv + v^2 dm = c^2 dm$$

Using this, equation (1) becomes

$$K = \int_{m_0}^{m} c^2 dm$$

where m_0 is the mass when velocity is zero and m is the mass when velocity is v

$$K = c^2 [m]_{m_0}^{m} = mc^2 - m_0 c^2$$

or $$K = (m - m_0)c^2.$$

It shows that change in mass is related to kinetic energy. i.e., Kinetic energy is defined as the product of increase in its mass and c^2. When the body is at rest $K = 0$.

i.e., $$mc^2 - m_0 c^2 = 0.$$

This shows that each term on L.H.S. should represent energy according to the principle of homogeneity of dimensions. Since m_0c^2 does not involve velocity of the body it should give energy associated with the particle at rest. Therefore m_0c^2 should be the internal energy of particle.

$\therefore$ Total energy of the particle $E = K + m_0c^2$

i.e., $$E = mc^2 - m_0c^2 + m_0c^2$$

$$E = mc^2$$

This shows that mass and energy are interconvertible and proves to be universal. The best examples of conversion of mass into energy are the nuclear fusion, nuclear reaction processes and phenomenon of pair annihilation. The best example of conversion of energy into mass is the phenomenon of pair production.

Example 29

Calculate the energy equivalent of 1kg of coal.

Solution

$$m = 1\text{kg}$$

using $$E = mc^2 = 1\times(3\times10^8)^2 = 9\times10^{16}\text{ J}$$

Example 30

Prove that when $\frac{v}{c} \ll 1$, the relativistic kinetic energy becomes the classical one.

Solution

We have relativistic kinetic energy

$$K = mc^2 - m_0c^2, \text{ using } m = \frac{m_0}{\sqrt{1-v^2/c^2}}$$

$$K = \frac{m_0}{\sqrt{1-v^2/c^2}}c^2 - m_0c^2$$

$$K = m_0c^2\left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right)$$

$$K = m_0c^2\left\{\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} - 1\right\}, \text{ when } \frac{v}{c} << 1$$

$$\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} = 1+\frac{v^2}{2c^2}$$

$$\therefore \qquad K = m_0c^2\left(1+\frac{v^2}{2c^2}-1\right)$$

$$K = m_0c^2\cdot\frac{v^2}{2c^2} = \frac{1}{2}m_0v^2$$

This is classical kinetic energy.

Example 31

Find the velocity of a proton having kinetic energy 900 MeV

$$m_0 = 1\cdot 67\times 10^{-27}\,\text{kg}.$$

Solution

Kinetic energy,

$$K = 900\text{MeV} = 900\times 10^6\text{eV} = 9\times 10^8\times 1\cdot 6\times 10^{-19}\text{J}$$

$$= 9\times 1\cdot 6\times 10^{-11}\text{J} = 14.4\times 10^{-11}\text{J}$$

using $\qquad K = (m - m_0)c^2$

$$14\cdot 4\times 10^{-11} = (m - m_0)\,(3\times 10^8)^2$$

or $\qquad m - m_0 = \dfrac{14.4\times 10^{-11}}{9\times 10^{16}} = 1\cdot 6\times 10^{-27}\,\text{kg}.$

or $\qquad m = m_0 + 1\cdot 6\times 10^{-27} = 1\cdot 67\times 10^{-27} + 1\cdot 6\times 10^{-27} = 3\cdot 27\times 10^{-27}.$

Using $\qquad m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$

$$3\cdot 27\times 10^{-27} = \frac{1\cdot 67\times 10^{-27}}{\sqrt{1-v^2/c^2}}$$

$$\sqrt{1-v^2/c^2} = \frac{1\cdot 67}{3\cdot 27} = 0\cdot 51$$

Squaring,
$$1-\frac{v^2}{c^2} = 0\cdot 26$$

$$\frac{v^2}{c^2} = 1-0\cdot 26 = 0.74$$

or
$$\frac{v}{c} = 0\cdot 86$$

$$v = 0\cdot 86c = 0\cdot 86\times 3\times 10^8$$

$$v = 2\cdot 58\times 10^8\,ms^{-1}\cdot$$

Example 32

Calculate the increase in mass when 1kg is moving with a velocity of 0.9 c. Hence find the kinetic energy.

Solution

$$m_0 = 1kg;\; v = 0\cdot 9c$$

using
$$m = \frac{m_0}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-0\cdot 9^2}} = \frac{1}{\sqrt{1-0\cdot 81}}$$

$$m = \frac{1}{\sqrt{0\cdot 19}} = 2\cdot 294\,kg$$

$\therefore$ Increase in mass $= m - m_0$

$$= 2\cdot 294 - 1 = 1\cdot 294\,kg$$

$\therefore$ Kinetic energy $K = (m-m_0)c^2$

$$= 1\cdot 294\times (3\times 10^8)^2 = 1\cdot 294\times 9\times 10^{16}$$

$$= 1\cdot 165\times 10^{17}\,J\cdot$$

Relativistic energy and momentum in an inelastic collision

Inelastic collision in Newtonian mechanics

Consider two identical particle each of mass m, moving with same velocity in opposite directions. After collision they stick together.

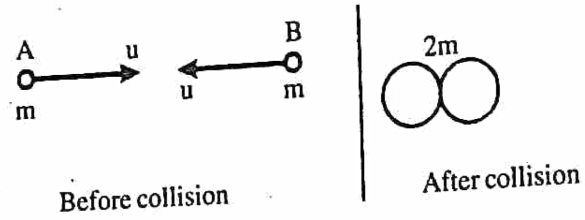


Figure 1.26

According to law of conservation of momentum

Momentum before collision = momentum after collision

$$mu - mu = 2mv$$
$$0 = 2mv$$

This implies that v = 0. i.e., after collision particles stick together and comes to rest

Total kinetic energy before collision

$$= \frac{1}{2}mu^2 + \frac{1}{2}mu^2 = mu^2$$

Total kinetic energy after collision

$$= \frac{1}{2} \cdot 2m0^2 = 0$$

This shows that during an inelastic collision energy is lost in the form of heat. This will not occur in relativity. We shall see to it.

Inelastic collision in relativity

Consider two identical particles each of rest mass m_0, moving with same velocity u in opposite directions. After collision particles stick together. In relativity we have two inertial frames S and S'. S is at rest and S' is moving uniformly. We analyse the collision with respect to S and S' separately.

In S-frame

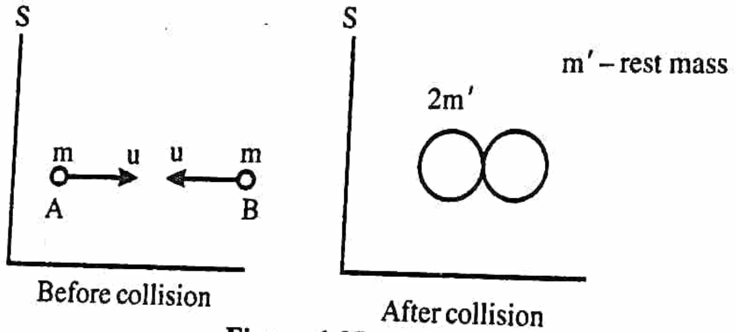


Figure 1.27

The total energy before collision = $mc^2 + mc^2$

$$= 2mc^2 = \frac{2m_0c^2}{\sqrt{1-\frac{u^2}{c^2}}}$$

The total energy after collision $= 2m'c^2$

As no external work has been done on the particles

Total energy before collision = Total energy after collision

i.e.,
$$\frac{2m_0c^2}{\sqrt{1-\frac{u^2}{c^2}}} = 2m'c^2 \qquad .....(1)$$

or
$$m' = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$$

i.e.,
$$m' > m_0$$

This shows that final rest mass (m') is greater than the initial rest mass. Unlike in Newtonian mechanics, in relativity the energy is not lost as heat but used to increase the mass according to mass-energy relation. In other words we can say that in relativity total energy is always conserved.

In S′ - frame

Let u be the velocity with which S' frame is moving along positive x-direction.

∴ The velocity of the particle A with respect to S' frame = 0

The velocity of the particle B with respect to S' frame

$$v = \frac{u+u}{1+\frac{uu}{c^2}} = \frac{2u}{1+\frac{u^2}{c^2}}$$

After collision the particles stick together and move with velocity u with respect to S' - frame. Situations are shown in figure below.

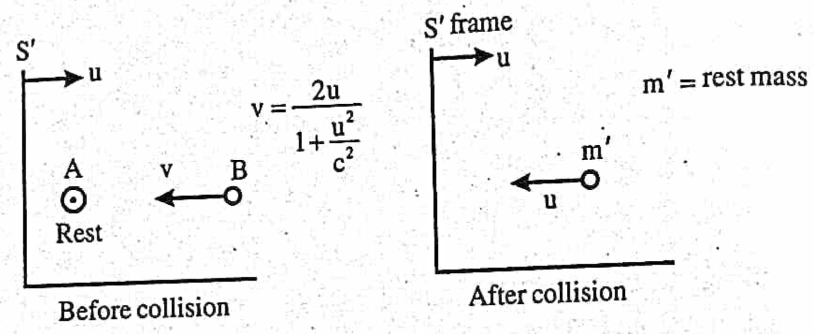


Figure 1.28

According to conservation of momentum

Momentum before collision = Momentum after collision

$$mv = \frac{2m'u}{\sqrt{1-u^2/c^2}}$$

or

$$\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2m'u}{\sqrt{1-\frac{u^2}{c^2}}} \qquad \text{..... (2)}$$

$$1-\frac{v^2}{c^2} = 1-\frac{(2u)^2}{\left(1+\frac{u^2}{c^2}\right)^2 c^2} = \frac{\left(1+\frac{u^2}{c^2}\right)^2 c^2 - 4u^2}{\left(1+\frac{u^2}{c^2}\right)^2 c^2}$$

$$= \frac{\left((1+2\frac{u^2}{c^2}+\frac{u^4}{c^4}\right)c^2 - 4u^2}{\left(1+\frac{u^2}{c^2}\right)^2 c^2}$$

$$= c^2 \frac{\left(1+\frac{2u^2}{c^2}+\frac{u^4}{c^4}-\frac{4u^2}{c^2}\right)}{\left(1+\frac{u^2}{c^2}\right)^2 c^2}$$

$$1-\frac{v^2}{c^2}=\frac{1-2\frac{u^2}{c^2}+\frac{u^4}{c^4}}{\left(1+\frac{u^2}{c^2}\right)^2}=\frac{\left(1-\frac{u^2}{c^2}\right)^2}{\left(1+\frac{u^2}{c^2}\right)^2} \qquad \text{..... (3)}$$

$$\therefore \qquad \frac{v}{\sqrt{1-\frac{v^2}{c^2}}}=\left(\frac{2u}{1+\frac{u^2}{c^2}}\right)\frac{\left(1+\frac{u^2}{c^2}\right)}{\left(1-\frac{u^2}{c^2}\right)}=\frac{2u}{1-\frac{u^2}{c^2}}$$

put this in equation (2), we get

$$m_0\frac{2u}{1-\frac{u^2}{c^2}}=\frac{2m'u}{\sqrt{1-\frac{u^2}{c^2}}}$$

or

$$\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}=m' \qquad \text{..... (4)}$$

we get back the same result in the S-frame. This shows that S and S' frames are identical.

Now we apply law of conservation of energy.

Total energy before collision = Total energy after collision

i.e.,

$$m_0c^2+\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{2m'c^2}{\sqrt{1-\frac{u^2}{c^2}}}$$

or

$$m_0+\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{2m'}{\sqrt{1-\frac{u^2}{c^2}}}$$

substituting for $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ from equation (3) we get

$$m_0 + \frac{m_0\left(1+\frac{u^2}{c^2}\right)}{\left(1-\frac{u^2}{c^2}\right)} = \frac{2m'}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$\frac{m_0\left(1-\frac{u^2}{c^2}\right)+m_0\left(1+\frac{u^2}{c^2}\right)}{1-\frac{u^2}{c^2}} = \frac{2m'}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$\frac{2m_0}{\sqrt{1-\frac{u^2}{c^2}}} = 2m'$$

or $m' = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$, we got back again the same result. Thus establishes the fact that S and S′ frame are identical and they yield the same results.

Example 33

Two particles of rest mass m_0 approach each other with equal and opposite velocity u in the laboratory frame. What is the total energy of one particle as measured in the rest frame of the other $u = \frac{c}{\sqrt{2}}$

Solution

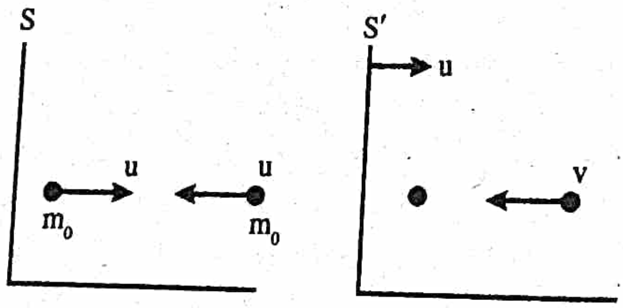


Figure 1.29

From the figure $v = \dfrac{u+u}{1+\dfrac{uu}{c^2}} = \dfrac{2u}{1+\dfrac{u^2}{c^2}}$ .....(1)

$$E' = mc^2 = \gamma m_0 c^2$$

Where $\gamma = \dfrac{1}{\sqrt{1-v^2/c^2}}$

$$\frac{u}{c} = \frac{1}{\sqrt{2}} \text{ given } \therefore \frac{u^2}{c^2} = \frac{1}{2}$$

Put this in equation (1) we get

$$v = \frac{2 \cdot c}{\sqrt{2}\left(1+\frac{1}{2}\right)} = 2\frac{\sqrt{2}}{3}c$$

$$1-\frac{v^2}{c^2} = 1-\left(2\frac{\sqrt{2}}{3}\right)^2 = 1-\frac{8}{9} = \frac{1}{9}$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{\frac{1}{9}}} = 3$$

So $E' = 3m_0c^2$

Example 34

A particle of rest mass m and speed v collides and sticks to a stationary particle of mass M. What is the final speed of the composite system.

Solution

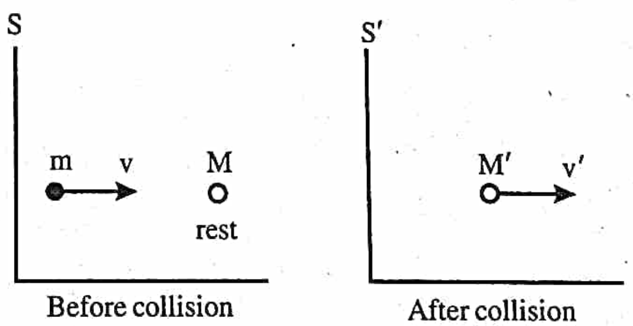


Figure 1.30

From momentum conservation

$$\frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{M'\,v'}{\sqrt{1-\frac{v'^2}{c^2}}} \qquad \text{.....(1)}$$

From energy conservation

$$\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} + Mc^2 = \frac{M'\,c^2}{\sqrt{1-\frac{v'^2}{c^2}}} \qquad \text{.....(2)}$$

$\frac{Eq(1)}{Eq(2)}$ gives
$$\frac{\frac{mv}{\sqrt{1-v^2/c^2}}}{\frac{mc^2}{\sqrt{1-v^2/c^2}}+Mc^2} = \frac{v'}{c^2}$$

Take $\frac{1}{\sqrt{1-v^2/c^2}} = \gamma \qquad \frac{\gamma mv}{\gamma m + M} = v'$

Example 35

A particle of rest mass m_0 and kinetic energy of $6m_0c^2$ strikes and sticks to an identical particle at rest. What is the rest mass of the resultant particle.

Solution

We have $\quad K = (m - m_0)c^2,\ K = 6m_0c^2$ gives

$$6m_0c^2 = (m - m_0)c^2$$

$$7m_0 = m \qquad \text{.....(0)}$$

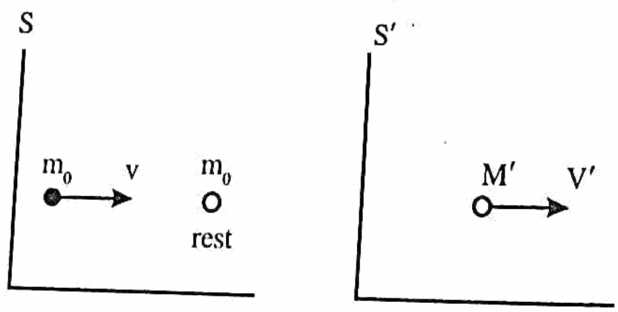


Figure 1.31

From momentum conservation

$$7m_0v = \frac{M' v'}{\sqrt{1-\frac{v'^2}{c^2}}} \qquad .....(1)$$

From energy conservation

$$7m_0c^2 + m_0c^2 = \frac{M' c^2}{\sqrt{1-\frac{v'^2}{c^2}}} \qquad .....(2)$$

$\frac{\text{eq 1}}{\text{eq 2}}$ gives $\frac{7}{8} v = v'$ .....(3)

From eq (0) we have $7m_0 = \frac{m_0}{\sqrt{1-v^2/c^2}}$

or $\sqrt{1-\frac{v^2}{c^2}} = \frac{1}{7}$ squaring

$$1-\frac{v^2}{c^2} = \frac{1}{49}$$

$$\frac{v^2}{c^2} = 1-\frac{1}{49} = \frac{48}{49}$$

$$\frac{v}{c} = \frac{\sqrt{48}}{7}$$

$$\therefore \quad v' = \frac{7}{8}\frac{\sqrt{48}}{7}c = \frac{\sqrt{48}}{8}c$$

$$\frac{v'}{c} = \frac{\sqrt{48}}{8}$$

$$\frac{v'^2}{c^2} = \frac{48}{64}$$

$$1-\frac{v'^2}{c^2}=1-\frac{48}{64}=\frac{16}{64}$$

$$\sqrt{1-\frac{v'^2}{c^2}}=\frac{4}{8}=\frac{1}{2}$$

Put the value of v' and $\sqrt{1-\frac{v'^2}{c^2}}$ in equation (1), we get

$$7m_0v=M'\frac{7}{8}v$$

$$\therefore \quad M'=4m_0.$$

The equivalence of mass and energy

The relativistic mass energy relation was experimentally confirmed by Cockcroft and Walton in 1932 by designing a high energy proton accelerator. The accelerator consists of mainly 4 parts.

1. A source of protons

Protons are produced by supplying an electrical discharge to hydrogen gas.

2. A power supply and a mechanism to boost the voltage of power supply

They used a power supply with a voltage of 150 kV. By using a simple electrical circuit consisting of capacitors and rectifiers, they boosted the voltage of the power supply step by step to 600 kV.

3. A target

They used ^7Li as the target

4. A zinc sulphide fluorescent screen construction and working

Protons produced were accelerated in vacuum step by step by the applied high voltage. When they attain maximum energy, they are allowed fall on the lithium target kept at 45°. The fluorescent screen is kept in front of the target as shown in figure 1.32.

They found that the fluorescent screen emitted occasional flashes. By various tests they determined that the flushes were due to α - particles. They interpreted this as follows. The ^7Li captures a proton and the resulting nucleus of mass 8 immediately disintegrates into alpha particles.

i.e.,

$$^1H+{}^7Li\rightarrow{}^4He+{}^4He$$

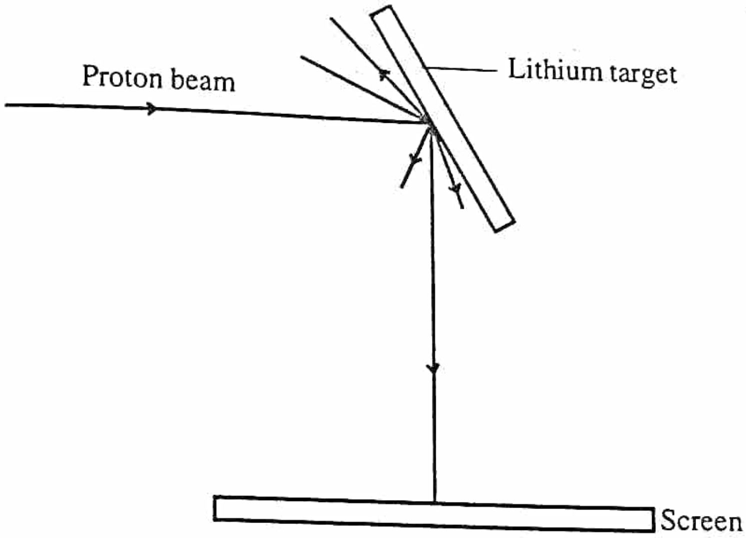


Figure 1.32

Writing mass-energy equation for the above reaction, we get

Kinetic energy of ^1H + Rest energy of ^1H and ^7L_i

= Kinetic energy of $2\,^4He$ + rest energy of $2\,^4He$

i.e $K(^1H) + M_p c^2 + M_{Li} c^2 = 2K(^4He) + 2M_\alpha c^2$

Rewriting this equation as

$(M_P + M_{Li} - 2M_\alpha)c^2 = 2K(^4He) - K(^1H)$

symbolically, we can write

$$\Delta Mc^2 = K \qquad \text{Mass – energy equation}$$

Cockcroft and Walton obtained the value $K = 17.2\,MeV$ from their experiment.

To get ΔMc^2, they substituted the values of M_p, M_{Li} and M_α

$$M_p = 1.0072 \text{ amu}$$

$$M_{Li} = 7.0104 \pm 0.0030 \text{ amu}$$

$$M_\alpha = 4.0011 \text{ amu}$$

$$\therefore \quad \Delta M = (1.0072 + 7.0104 \pm 0.0030) - 2 \times 4.0011)$$

$$= (0.0154 \pm 0.0030)$$

Using 1 amu $= 931\,\text{MeV}$

$$\Delta Mc^2 = (14.34 \pm 2.79)\text{MeV}$$

The difference between K and ΔMc^2 is $= (17.2 - 14.34)$ MeV $= 2.86$ MeV. This is only slightly larger than the allowed experimental uncertainty 2.79. From this they almost confirmed the equivalence of mass and energy.

Momentum - Energy relation

If a body of mass m moving with a velocity v its momentum

$$p = mv \qquad \text{..... (1)}$$

and its energy

$$E = mc^2 \qquad \text{..... (2)}$$

Using $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$ squaring on both sides and rearranging we get

$$m^2 = \frac{m_0^2}{1 - v^2/c^2}$$

$$m^2\left(1 - \frac{v^2}{c^2}\right) = m_0^2$$

$$m^2\left(\frac{c^2 - v^2}{c^2}\right) = m_0^2$$

$$m^2(c^2 - v^2) = m_0^2 c^2$$

$$m^2 c^2 - m^2 v^2 = m_0^2 c^2$$

$$m^2 c^2 = m^2 v^2 + m_0^2 c^2$$

Multiplying throughout by c^2, we get

$$m^2 c^4 = m^2 v^2 c^2 + m_0^2 c^4$$

Using equations (1) and (2) we get

$$E^2 = p^2 c^2 + m_0^2 c^4$$

$$E = \sqrt{p^2 c^2 + m_0^2 c^4}$$

This is the energy-momentum relation.

Massless particles

In the case of a photon or neutrino $m_0 = 0$

$$\therefore \quad E = pc$$

using equations (1) and (2), we have

$$mc^2 = m\,v\,c$$

$$v = c$$

This shows that the velocity of a particle of zero rest mass is c.

Remember that only rest mass of the photon is zero. Using $E = mc^2$ and $E = h\upsilon$

We have $$mc^2 = h\upsilon$$

or $$m = \frac{h\upsilon}{c^2}$$

This is the mass equivalent of photon.

It shows that in relativity a photon of energy E has a mass $\frac{h\upsilon}{c^2}$. Since this massless particle interacts with charged particles like electrons, protons etc., they can be easily detectable. Similarly the particles neutrino and graviton are massless particles though they interact with matter weakly they cannot be detected easily. So far neutrinos have been detected but not gravitons.

Tachyons

In nature we find particles like electrons, protons etc. for which $m_0 > 0$. From this an interesting proposal has been made by E.C.G. Sudarsan (an Indian scientist) and others that it in nature particles with $m_0 > 0$ and $m_0 = 0$ exist then particles with $m_0 < 0$ should also exist. Such particles can have velocity greater than c.

Using $$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

If $v > c, \sqrt{1 - v^2/c^2}$ is imaginary. If m_0 is taken to be imaginary. The imaginary in the numerator and the imaginary in the denominator will cutoff there by making m a real quantity. The rest mass imaginary means that this particle cannot have a rest position. The rest mass m_o imaginary does not matter provided the experimentally observable entities E and p are real and positive. This condition is satisfied provided $p^2c^2 > m_0^2c^4$ (From $E^2 = p^2c^2 + m_0^2c^4$). Also since $E = \gamma mc^2$ and $p = \gamma m_0 v$, E and p can be real only if γ is imaginary. Physically this means that particle is moving with a velocity greater than c. These particles are called Tachyons. This name is derived from Greek word Tachyons meaning swift. Several attempts have been made to observe these particles but so far none of them has been successful.

Example 36

Show that the particles move with velocity equal to that of light have zero rest mass

Solution

From Energy-momentum relation, we have

$$E = \sqrt{p^2c^2 + m_0^2c^4}$$

or $\quad E^2 = p^2c^2 + m_0^2c^4$ using $E = mc^2$ and $p = mv$

$$m^2c^4 = m^2v^2c^2 + m_0^2c^4$$

But $\quad v = c$ given, then

$$m^2c^4 = m^2c^4 + m_0^2c^4$$

or $\quad m_0^2c^4 = 0 \Rightarrow m_0 = 0.$

Example 37

Calculate the rest mass of a particle whose momentum is $130\,\mathrm{MeV/c}$ when its kinetic energy is $50\,\mathrm{MeV}$.

Solution

$$p = 130\frac{\mathrm{MeV}}{c} = \frac{130\times10^6\times1.6\times10^{-19}}{3\times10^8}\,\mathrm{kgms^{-1}}$$

$$K = 50\ \mathrm{MeV} = 50\times10^6\times1.6\times10^{-19}\,\mathrm{J}$$

The kinetic energy K and the total energy are related by

$$E = K + m_0c^2$$

and also we have

$$E = \sqrt{p^2c^2 + m_0^2c^4}$$

equating the two equations, we get

$K + m_0c^2 = \sqrt{p^2c^2 + m_0^2c^4}$ squaring on both sides.

$$K^2 + m_0^2c^4 + 2Km_0c^2 = p^2c^2 + m_0^2c^4$$

$$2Km_0c^2 = p^2c^2 - K^2$$

$$m_0 = \frac{p^2c^2 - K^2}{2Kc^2}$$

$$= \frac{(130\times10^6\times1\cdot6\times10^{-19})^2\,(3\times10^8)^2 - (50\times10^6\times1\cdot6\times10^{-19})^2}{2\times50\times10^6\times1\cdot6\times10^{-19}\,(3\times10^8)^2}$$

$$= 2\cdot56\times10^{-28}\,\text{kg.}$$

The photoelectric effect

The phenomenon of photoelectric effect had been studied extensively in the quantum mechanics paper. Photoelectric effect confirmed that light possesses particle nature. But in relativity Michelson interferometer experiment confirmed that light has wave nature. Einstein's energy relation $mc^2 = h\upsilon$ provides a link between the particle and wave nature of light.

Radiation pressure of light

According to Maxwell's electro magnetic theory, light wave carries momentum. This momentum will be transferred to the surface when it is absorbed or reflected. It is due to the change in moment, when light falls on a surface, the surface experiences a pressure called radiation pressure. The calculation of radiation pressure on the basis of wave theory is complicated, but with photos picture it is very simple.

Calculation of radiation pressure on the basis of photon picture

Consider a stream of photons striking a perfectly reflecting mirror normally. After reflection photons move with the same speed.

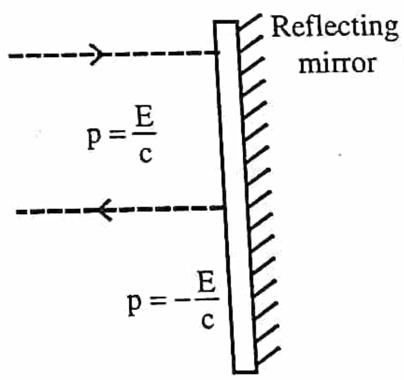


Figure 1.33

The initial momentum of each photon $p = \frac{E}{c}$

Final momentum (after reflection) of each photon,

$$p = -\frac{E}{c}$$

$\therefore$ Change in momentum of the photon $= \frac{E}{c} - \frac{-E}{c} = \frac{2E}{c}$

If there are n photons stricking the surface per unit area in unit time

The total change in momentum in unit time per unit area $= \frac{2nE}{c}$

The change in momentum in unit time is called force F.

Thus Force per unit area $= \frac{2nE}{c}$

The force per unit area is called pressure P

$$\therefore \quad P = \frac{2nE}{c}$$

nE is the total energy per unit area per second is called the intensity of light I

$$\text{Thus } P = \frac{2I}{c}$$

This is the expression for radiation pressure when light reflects from a reflector.

Similarly the radiation pressure on an absorbing surface $P = \frac{I}{c}$

The average intensity of sunlight falling on the earth surface at normal incidence is calculated to be $1400\,\text{Wm}^{-2}$.

$\therefore$ The radiation pressure on a mirror due to sunlight $= \frac{2I}{c} = \frac{2 \times 1400}{3 \times 10^8}$

$$= 9.33 \times 10^{-6}\,\text{Nm}^{-2}$$

This pressure is very small when comparing to atmospheric pressure $10^5\,\text{Nm}^{-2}$. However on the cosmic scale the radiation pressure is very large, it helps keep stars from collapsing under their own gravitation forces. The gravitational force of a star is acting inward and the radiation pressure is acting outward. So long as when these two forces balance each other the star lives as a youngster one.

Another important aspect of photons is that unlike classical particles photons can be created and destroyed. In other words photon number is not conserved in nature. The absorption of light by matter leads to destruction of photon where as emission of radiation creates photons. Nevertheless, the law of conservation of momentum and energy are generalised with in the frame work of relativity. This will provide us a very powerful tool to treat processes involving photon without a detailed knowledge of interaction. One illustrative example is given below.

The photon picture of the Doppler effect

We already discussed the Doppler effect of light by considering light as a wave. This time we are going to discuss Doppler effect of light by considering light as a particle (photon).

Consider an atom with rest mass m_0 at rest. If the atom emits a photon of energy $h\upsilon$ the mass of the atom becomes m_0'. From the conservation of energy we have

$$m_0c^2 = m_0'c^2 + h\upsilon$$

i.e., $$m_0'c^2 = m_0c^2 - h\upsilon \quad \text{.......(1)}$$

Now suppose that the atom moves freely with a velocity v, before emitting the photon.

Before emitting the photon

Energy of the atom, $E = \dfrac{m_0c^2}{\sqrt{1-v^2/c^2}}$

Momentum of the atom $p = \dfrac{m_0v}{\sqrt{1-v^2/c^2}}$

After emitting the photon of energy $h\upsilon'$, the atom has velocity v', rest mass m_0', energy E' and momentum p'.

According to conservation momentum, we have

$$p = p' + \frac{h\upsilon'}{c}$$

or $$pc - h\upsilon' = p'c \quad \text{.....(2)}$$

According to conservation of energy, we have

$$E = E' + h\upsilon'$$

or $$E - h\upsilon' = E' \quad \text{......(3)}$$

squaring equations(2) and (3) and subtracting we get

$$(E - h\upsilon')^2 - (pc - h\upsilon')^2 = E'^2 - (p'c)^2$$

From the energy-momentum relation, we have

$$E^2 = p^2c^2 + m_0^2c^4$$

so

$$E'^2 = p'^2c^2 + m_0'^2c^4$$

or

$$E'^2 - p'^2c^2 = m_0'^2c^4$$

Then above equation becomes

$$(E - h\upsilon')^2 - (pc - h\upsilon')^2 = m_0'^2c^4$$

substituting for $m_0'^2c^2$ from eq. (1), yields

$$(E - h\upsilon')^2 - (pc - h\upsilon')^2 = (m_0c^2 - h\upsilon)^2$$

$$E^2 - 2Eh\upsilon' + h^2\upsilon'^2 - p^2c^2 + 2pch\upsilon' - h^2\upsilon'^2$$

$$= m_0^2c^4 - 2m_0c^2h\upsilon + h^2\upsilon^2$$

substituting for $E^2 - p^2c^2 = m_0^2c^4$ on the L.H.S and cut likes terms on both sides, we get $-2Eh\upsilon' + 2pch\upsilon' = -2m_0c^2h\upsilon + h^2\upsilon^2$

$$2h\upsilon'(-E + pc) = -h\upsilon(2m_0c^2 - h\upsilon)$$

or

$$\upsilon' = \upsilon\,\frac{(2m_0c^2 - h\upsilon)}{2(E - pc)}$$

Now we evaluate $E - pc$

$$E - pc = \frac{m_0c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{m_0vc}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$E - pc = \frac{m_0c^2}{\sqrt{1 - \frac{v^2}{c^2}}}\left(1 - \frac{v}{c}\right) = m_0c^2\sqrt{\frac{\left(1 - \frac{v}{c}\right)}{\left(1 + \frac{v}{c}\right)}}$$

$$\therefore \quad \upsilon' = \upsilon\, 2m_0c^2\left(1-\frac{h\upsilon}{2m_0c^2}\right)\cdot\frac{1}{2m_0c^2}\sqrt{\frac{\left(1+\frac{v}{c}\right)}{\left(1-\frac{v}{c}\right)}}$$

$$\upsilon' = \upsilon\left(1-\frac{h\upsilon}{2m_0c^2}\right)\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

For massive sources the term $\frac{h\upsilon}{2m_0c^2}$ is negligible.

$$\therefore \quad \upsilon' = \upsilon\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

A result which is exactly in agreement with our old result. This shows that the wave nature and the particle nature of light predict the same results.

Does light travel at the velocity of light

The question does light travel at the velocity of light may seem to be rhetorical. But answering this question is not as simple as we think of. We know that the entire edifice of relativity is built upon on two postulates where velocity plays a crucial role. In relativity a universal character is attributed to the velocity of light i.e, the velocity of light in vacuum is a universal constant and is the same for the observer in all inertial frames. We can very well prove that there is only one such universal velocity in relativity.

To prove this we assume that there is a second universal velocity c^* emitted by a system or due to some phenomenon other than light. For example the velocity of gravitons or neutrinos or tachyons. We call this phenomenon as Γ.

Consider a light and a Γ - pulse emitted along the x-axis from the origin of a coordinate system (S frame) at $t = 0$

The x-coordinates of the pulses after a time t are

$$x_l = ct$$

$$x_\Gamma = c^* t$$

The relative velocity between the two pulses is given by

$$u = \frac{d}{dt}(x_\Gamma - x_l) = \frac{d}{dt}(c^* t - ct)$$

$$u = c^* - c$$

Now consider the same pulses in the S' frame moving with velocity v along the positive x-axis.

The x- coordinates of the pulses with respect to S' frame are

$$x'_l = ct'$$

$$x'_\Gamma = c^* t'$$

∴ The relative velocity of the two pulses is given by

$$u' = \frac{d}{dt'}(x'_\Gamma - x'_l) = \frac{d}{dt}(c^* t' - ct')$$

$$u' = c^* - c$$

From the velocity addition theorem

we have $$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

For light pulse $$c' = \frac{c - v}{1 - \frac{cv}{c^2}} = \frac{c - v}{1 - \frac{v}{c}} = c$$

For Γ - pulse $$(c^*)' = \frac{c^* - v}{1 - \frac{c^* v}{c^2}}$$

$c' = c$ implies that the velocity of light in S frame and S' frame is the same there by attaining a universal character in accordance with relativity. We also proved that

$(c^*)' \neq c^*$ implies that Γ - pulse is not universal in its character. Moreover using Lorentz transformation we can very well write

$$u' = (c^*)' - c$$

if c^* and c are universal.

or
$$u' = \frac{c^* - v}{1 - \frac{c^* v}{c^2}} - c$$

This result disagrees with our former result.

$$u' = c^* - c$$

For these results to be agreeable only when $c^* = c$

In this case $u = c^* - c = 0$

and
$$u' = \frac{c - v}{1 - \frac{cv}{c^2}} - c = 0$$

$c = c^*$ implies that there is only one universal velocity i.e, the velocity of light. We conclude that light travels at the universal velocity. We could also see that when we try to incorporate a second universal velocity the entire edifice of relativity collapsed. In other words special theory of relativity cannot accommodate more than one universal velocity.

Scientific community is still investigating a second universal velocity moving faster than light. If it is found we have to modify special theory of relativity in order to accommodate the newly discovered.

The rest mass of the photon

We found that within the frame work of relativity the rest mass of the photon is zero and it moved with velocity of light of the photon has a non-zero rest mass, its velocity would be less than c.

If m_p is the rest mass of the photon, then its energy

$$E = \gamma m_p c^2$$

If we assume that $E = h\upsilon$ is valid, the above equation can be written as

$$h\upsilon = \frac{m_p c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

squaring on both sides, we get

$$(h\upsilon)^2 = \frac{(m_p c^2)^2}{1-\frac{v^2}{c^2}}$$

If υ_0 be the characteristic frequency of photon, then $m_p c^2 = h\upsilon_0$

$$\therefore \quad (h\upsilon)^2 = \frac{(h\upsilon_0)^2}{1-\frac{v^2}{c^2}}$$

$$\upsilon^2 = \frac{\upsilon_0^{\,2}}{1-\frac{v^2}{c^2}}$$

or $$\upsilon^2\left(1-\frac{v^2}{c^2}\right) = \upsilon_0^{\,2}$$

$$1-\frac{v^2}{c^2} = \frac{\upsilon_0^{\,2}}{\upsilon^2}$$

$$\therefore \quad \frac{v^2}{c^2} = 1-\frac{\upsilon_0^{\,2}}{\upsilon^2}$$

If $\upsilon_0 = 0$, then $v = c$, otherwise v depends on frequency. This happen when light passes through dispersive medium such as water, glass etc. This phenomenon is called dispersion. Our experimental challenge is to test whether or not vacuum (empty place) exhibits dispersion. If it is proved we will be forced to put a limit on the rest mass of the photon. So far the frequency dependence of light in vacuum is not confirmed. So the rest mass of the photon can be taken to be zero.

Light from a pulsar

Pulsar is the abbreviated form of a pulsating star. Pulsars emit light signals at regular intervals of time. The time intervals vary from milliseconds to 10 seconds. Depending upon the nature of light signals emitted by pulsars we have several types of pulsars such as radio pulsars, x-ray pulsars, gamma ray pulsars etc. The first pulsar was detected by Joselyn Bell and Anthony Hewish in the year 1967. It is named as PSR1919+21. The time interval of signals of this pulsar was 1.337s. For this discovery Anthony Hewish was awarded the Nobel prize in physics for the year 1974. The peculiar properties of light signals emitted by pulsars attracted the scientific community. As a result of this more than 1000 pulsars have been detected so far. The most interesting pulsar is the one that we found in the crab nebula called crab pulsar.

It emits signals of time intervals 0.033s. It is due to high precision of time interval of emission of signals, crab pulsar is considered as the superclock of the universe. Crab pulsar emits signals in the optical, x-ray and radio frequency regions. As the pulses are quite sharp their arrival time can be measured to an accuracy of micro seconds. it is known that light from the pulsars at different optical wavelengths arrives simultaneously within the experimental resolving time. We can use these facts to put a limit on the rest mass of the photon.

Signal from the crab pulsar takes 5000 years, to reach on earth. Suppose that signals at two different frequencies travel with a small difference in velocity Δv and obviously arrive at slightly different times t and $t + \Delta t$ on earth.

Using, time $= \dfrac{\text{distance}}{\text{velocity}}$

$$t = \frac{l}{v}$$

Where l is the distance between the crab pulsar and the earth.

or $$v = \frac{l}{t} \qquad \text{..... (1)}$$

$\therefore$ $$\Delta v = -\frac{l}{t^2}\Delta t \qquad \text{..... (2)}$$

$\frac{\text{eq2}}{\text{eq1}}$ gives $$\frac{\Delta v}{v} = -\frac{\Delta t}{t}$$

Actually no such velocity difference has been observed. But by estimating the sensitivity of the experiment we can set an upper limit to $\Delta v \cdot \Delta t$ can be measured to an accuracy of about 2 milliseconds.

i.e., $$\Delta t = 2 \times 10^{-3} s$$

we know that $$t = 5000 \text{ years}$$

$$t = 5000 \times 3.15 \times 10^{7} s$$

(1 year $\simeq 3.15\times10^7$s) $\quad t \simeq 1.58\times10^{11}$s

$$\therefore \quad \left|\frac{\Delta v}{v}\right| = \left|\frac{\Delta t}{t}\right| \leq \frac{2\times10^{-3}}{1.5\times10^{11}}$$

or
$$\left|\frac{\Delta v}{v}\right| \simeq 10^{-14}$$

If we take $v \simeq c$

$$\left|\frac{\Delta v}{c}\right| \simeq 10^{-14}$$

Recall our earlier result

$$\frac{v^2}{c^2} = 1 - \frac{\upsilon_0^2}{\upsilon^2}$$

For the two signal frequencies υ_1 and υ_2 and the velocities v_1 and v_2, we get

$$\frac{v_1^2}{c^2} = 1 - \frac{\upsilon_0^2}{\upsilon_1^2}$$

and
$$\frac{v_2^2}{c^2} = 1 - \frac{\upsilon_0^2}{\upsilon_2^2}$$

subtracting, we get

$$\frac{v_1^2 - v_2^2}{c^2} = \upsilon_0^2\left(\frac{1}{\upsilon_2^2} - \frac{1}{\upsilon_1^2}\right)$$

or
$$\frac{(v_1 + v_2)(v_1 - v_2)}{c^2} = \upsilon_0^2\left(\frac{1}{\upsilon_2^2} - \frac{1}{\upsilon_1^2}\right)$$

Take $v_1 + v_2 \simeq 2c$ and $v_1 - v_2 = \Delta v$, we have

$$2\frac{\Delta v}{c} = \upsilon_0^2\left(\frac{1}{\upsilon_2^2} - \frac{1}{\upsilon_1^2}\right)$$

Take $\quad \upsilon_1 = 8\times10^{14}$Hz (blue light)

and $\quad \upsilon_2 = 5\times10^{14}$Hz (red light)

$$2\frac{\Delta v}{c} = \frac{\upsilon_0^2}{10^{28}}\left(\frac{1}{5^2} - \frac{1}{8^2}\right) = 2.44 \times 10^{-30}\upsilon_0^2$$

Using the limit $\frac{\Delta v}{c} = 10^{-14}$, we have

$$2 \times 10^{-14} = 2.44 \times 10^{-30}\upsilon_0^2$$

or $$\upsilon_0^2 = \frac{2 \times 10^{-14}}{2.4 \times 10^{-30}} = 0.833 \times 10^{16}$$

$$\upsilon_0^2 < 10^{16}$$

$$\upsilon_0 < 10^8$$

Putting this value in the expression for rest mass of the photon

$$m_p = \frac{h\upsilon_0}{c^2} = \frac{6.6 \times 10^{-34} \times < 10^8}{9 \times 10^{16}}$$

$$m_p < 0.733 \times 10^{-42}$$

That is the limiting mass of the photon is 10^{-42} kg. If we increase the experimental accuracy $\frac{\Delta v}{c}$, the limiting rest mass of the photon will also increase.

IMPORTANT FORMULAE

1. Michelson interferometer:

   a) Displacement of interference fringes, $\delta = \frac{2Lv^2}{c^2}$

   (b) Number of fringe shifts = $\frac{\delta}{\lambda}$

   (c) Fringe shift in water = $\frac{4n^2 l}{\lambda c} v_w f$, where $f = 1 - \frac{1}{n^2}$

2. Galilean transformations:

   a) Coordinate transformation equations $x' = x - vt$, $y' = y$ and $t' = t$

b) Velocity transformation equations $u'_x = u_x - v$, $u'_y = u_y$ and $u'_z = u_z$

c) Acceleration transformation equations $a'_x = a_x$, $a'_y = a_y$, and $a'_z = a_z$

3. Lorentz transformation equations:

$$x' = \gamma(x - vt) \qquad x = \gamma(x' + vt')$$

$$y' = y \qquad y = y'$$

$$z' = z \qquad z = z'$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \qquad t = \gamma\left(t' + \frac{vx'}{c^2}\right) \quad \text{where } \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

4. Space like and time like intervals.

$$L' = \gamma(L - vT)$$

and $T' = \gamma\left(T - \frac{v}{c^2}L\right)$

For space like interval

$L > cT$, L' is positive and T' can be positive, zero or negative.

For time like interval

$L < cT$, T' is positive and L' can be positive, negative or zero.

5. Lorentz length contraction formula:

$$L = L_0\sqrt{1 - v^2/c^2}, \quad L < L_0.$$

6. The orientation of a moving rod:

$$L = L'\left(1 - \sqrt{1 - \frac{v^2}{c^2}}\cos\theta'\right)^{\frac{1}{2}}$$

$$\theta = \tan^{-1}(\gamma\tan\theta'), \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$L < L', \quad \theta > \theta'$$

7. Atomic clock correction:

$$\frac{\delta\upsilon}{\upsilon_0} = -\frac{3kT}{2mc^2}$$

8. Time dilation

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}} \qquad \Delta t > \Delta t'$$

9. Transformation of velocities:

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \qquad u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}}$$

$$u'_y = \frac{u_y\sqrt{1 - v^2/c^2}}{1 - \frac{u_x v}{c^2}} \qquad u_y = \frac{u'_y\sqrt{1 - v^2/c^2}}{1 + \frac{u'_x v}{c^2}}$$

$$u'_z = \frac{u_z\sqrt{1 - v^2/c^2}}{1 - \frac{u_x v}{c^2}} \qquad u_z = \frac{u'_z\sqrt{1 - v^2/c^2}}{1 + \frac{u'_x v}{c^2}}$$

10. The velocity of light in moving water with respect to lab frame.

$$u = \frac{c}{n} + \left(1 - \frac{1}{n^2}\right)v$$

11. Doppler effect in sound:

$$\upsilon' = \upsilon\frac{1}{1 - \frac{v}{w}}$$ – Source is moving towards the observer

$$\upsilon' = \upsilon\left(1 + \frac{v}{w}\right)$$ – Observer is moving towards the source

12. Relativistic Doppler effect

$$\upsilon' = \upsilon\sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}$$ – Source is moving towards the observer

or $$\lambda' = \lambda\sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}$$

13. Doppler effect for an observer off the line motion

$$\upsilon' = \upsilon\frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v}{c}\cos\theta}$$ Source is moving towards the observer

14. Doppler navigation formula

$$r_b - r_a = -\lambda N_{ab}$$

15. Relativistic momentum

$$\vec{p} = m\vec{u} = \frac{m_0\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}$$

16. Variation of mass with velocity

$$m(v) = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

17. Mass-energy relation:

$$E = mc^2 = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}, \quad E = K + m_0c^2$$

18. Expression for relativistic kinetic energy

$$K = mc^2 - m_0c^2 = m_0c^2\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right)$$

19. Energy-momentum relation

$$E = \sqrt{p^2c^2 + m_0^2c^4}$$

20. Expression for radiation pressure

$$P = \frac{2nE}{c} = \frac{2I}{c}$$ - Light reflected from a mirror

$$P = \frac{I}{c}$$ - For an absorbing surface

21. Rest mass of the photon, $m_0 = 0$

Mass of the photon $m = \frac{h\upsilon}{c^2}$

22. Orientation of a light pulse with respect to S frame

$$\cos\theta = \frac{\cos\theta' + \frac{v}{c}}{1 + \frac{v}{c}\cos\theta'}$$

UNIVERSITY MODEL QUESTIONS

Section A

(*Answer questions in about* **two** *or* **three** *sentences*)

Short answer type questions

1. What was the aim of the Michelson-Morley experiment?
2. Draw a labelled diagram of Michelson-Morley experiment
3. Explain the negative result of Michelson - Morley experiment.
4. What is the use of compensating plate in Michelson - Morley experiment?
5. Explain the significance of Michelson - Morley experiment.
6. What are tht postulates of special theory of relativity?
7. Write down the Galilean transformation equations and explain the symbols used.
8. Obtain the Galilean transformation equations for velocity of a particle moving in space.
9. Obtain the Galilean acceleration transformation equations.
10. What is a Galilean invariant quantity?
11. Write down the Lorentz transformation equations and explain the symbols used.
12. What is meant by simultaneity?
13. What is meant by time like interval?
14. What is meant by space like interval?
15. Draw the graphical variation of $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$ with $\frac{v}{c}$.
16. What is Lorentz contraction in relativity?
17. What is meant by time dilation in relativity?
18. Define the following terms

    a) Proper frame b) Proper length c) Proper time
19. Write down the velocity transformation equations in relativity.
20. What is the effect on speed of light in a moving medium?
21. Define Doppler effect.
22. Write down the relativistic Doppler formula in terms of frequency and explain the symbols used.
23. Write down two applications of Doppler effect.

24. What is twin paradox?
25. Define relativistic momentum and energy.
26. How does mass vary with velocity
27. Write down the mass energy relation and explain the symbols used.
28. State the expressions for rest energy, kinetic energy and total energy of a relativistic particle.
29. Suppose the speed of light were infinite what would happen to the relativistic predictions of length contraction, time dilation and mass variation.
30. What is meant by Doppler navigation?
31. Write down the energy momentum relation and explain the symbols used.
32. Draw the graphical variation of mass with velocity.
33. Distinguish between elastic and inelastic collision in relativity.
34. Massless particles travel with speed of light. Justify.
35. What is meant by radiation pressure of light?
36. Write down expression for radiation pressure and explain the symbols used.
37. If a photon travels with a speed other than that of light, then what would be the rest mass of the photon
38. What is a pulsar? Name two of them.
39. What is the effect of radiation pressure in stars?
40. Why pulsars are considered as super clocks of the universe?

Section B

(Answer questions in about half a page to one page)

Paragraph / Problem

1. Show that in the non-relativistic limit the Lorentz transformations reduce to Galilean transformations.
2. Explain how time dilation was verified experimentally.
3. Show that when $\frac{v}{c} \ll 1$, , the relativistic kinetic energy becomes the classical one.
4. Show that addition of velocity to the velocity of light gives velocity of light.
5. Events that are simultaneous in one frame are not simultaneous in another reference frame, prove it.
6. Explain how velocity transformation was verified experimentally.
7. Explain how mass variation was verified experimentally.
8. Arrive at space like and time like intervals.
9. A rod of length L' in S' frame moving with a speed v. Find out its orientation in the lab frame.
10. What is the role of time dilation in atomic clocks?
11. Arrive at velocity addition theorem in relativity.

12. Deduce an expression for velocity of light in moving water with respect to lab frame.
13. How will you track a moving object by Doppler effect?
14. Explain the ageing of twins quantitatively.
15. Derive the relation $E = mc^2$
16. Derive the energy-momentum relation from mass variation with velocity.
17. Derive an expression for radiation pressure of light.

Problems

18. What will be the fringe shift in Michelson Morley experiment, if the effective length of each part is 6 m and wavelength of light used is 6000 Å ? Velocity of earth $3\times10^4\,ms^{-1}$.
[$\frac{1}{5}$of a fringe]
19. In Michelson-Morley experiment the length of the paths of the two beams is 11 metres each. The wavelength of light used is 6000Å. If the expected fringe shift is 0.4 fringes, calculate the velocity of earth relative to ether
[$v = 3\times10^4\,ms^{-1}$]
20. Show that for $v \ll c$, the Lorentz transformations will become Galilean transformations.
21. At what speed v, will the Galilean and Lorentz expressions for x differ by
a) 1% b) 50%
[$42\times10^8\,ms^{-1}$, $2\cdot235\times10^8\,ms^{-1}$.]
22. The length of a space craft is measured to be exactly $\frac{3}{4}$ of its proper length. What is its speed with respect to earth. Also find the dilation of spacecrafts unit time.
[a) $v = 0.66c = 1.98\times10^8\,ms^{-1}$ b) $\Delta t_0 = 0.75s$]
23. Let a meter scale be moving with a velocity half of that of light. What will be its length measured by a stationary observer.
[L = 0.866 m]
24. A rod of proper length 3m moves with a velocity 0.86 c in a direction making an angle 60° with its length. Find the apparent length.
[2.7495 m]
25. Find the speed of the space ship of every day spent on it may correspond to 2 days on the earth surface.
[$v = 2.598\times10^8\,ms^{-1}$]
26. At what speed should a clock be moved so that it may appear to lose 1 minute in each hour
[$v = 5.4\times10^7\,ms^{-1}$]
27. Calculate the energy equivalent of the rest mass of an electron.
[$E = 8.19\times10^{-14}\,J$]
28. Calculate the velocity of an electron having kinetic energy 1 MeV. $m_0 = 9\times10^{-31}\,kg$.
[$v = 2\cdot62\times10^8\,ms^{-1}$]
29. Calculate the energy released in MeV when a neutron decays into a proton and an electron. $M_n = 1\cdot6747\times10^{-27}\,kg$, $M_e = 9\times10^{-31}\,kg$.
[E = 0.7303 MeV]

30. An electron and a positron practically at rest come together and annihilate into other, producing two protons of equal energy. Find the energy and equivalent mass of each proton. $m_e = 9\times10^{-31}$ kg.

[energy of each proton $= 81\times10^{-15}$ J mass equivalent of each proton 9×10^{-31} kg]

31. Show that the rest mass of a particle of momentum p and kinetic energy K is $m_0 = \dfrac{p^2c^2 - K^2}{2Kc^2}$.

32. A particle with rest mass m_0 is moving with a velocity $\dfrac{c}{\sqrt{2}}$. Find out the momentum, kinetic energy and total energy of this particle. $\left[m_0c, (\sqrt{2}-1)\,m_0c^2, \sqrt{2}\,m_0c^2\right]$

33. The rest mass of an electron is 9.1×10^{-31} kg what will be its mass if it were moving with $\dfrac{4}{5}$th of the speed of light. [1.5×10^{-30} kg]

34. Suppose that the total mass of 1kg is transformed into energy, how large is this energy in kilowatt hours [2.5×10^{10}]

35. A certain quantity of ice at $0°$ C melts into water at ice at $0°$ C, in doing so gains 1 kg of mass. What was the initial mass. [2.68×10^{11} kg]

36. Show that $x^2 + y^2 + z^2 - c^2t^2$ is invariant under Lorentz transformation.

37. Derive $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$ starting from $E = mc^2$

38. Derive the mass transformation equation $m = \gamma m'\left(1 + \dfrac{u'_x v}{c^2}\right)$

39. Show that the differential operator

$$\Box^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$ is invariant under Lorentz transformation.

40. Show that the proper time $d\tau = dt\sqrt{1 - \dfrac{v^2}{c^2}}$ is an invariant quantity under Lorentz transformation.

41. What is the speed of π-mesons whose observed mean life is 2.5×10^{-7} s. The proper mean life of these π-mesons is 2.5×10^{-8} s. [$v = 0.995c$]

42. A beam of pions has velocity $v = 0.6c$. It has a half life of 1.8×10^{-8} s. Estimate the time taken by pions to decay to half their initial number. [2.25×10^{-8} s]

43. Find the shape of a circle at rest in a frame S when viewed from a frame S', when S' is moving with speed v along x-axis with respect to S. [ellipse]

44. Frame S' moves with velocity v relative to frame S. A bullet in frame S is fired with velocity u at an angle θ with respect to forward direction of motion. What is this angle as measured in S'. $\left[\tan\theta' = \tan\theta\sqrt{1-\frac{v^2}{c^2}}\left(1-\frac{v}{u\cos\theta}\right)\right]$

45. Two β-particles A and B travel in opposite directions each with a velocity 0.98 c. What is their relative velocity as observed by a stationary observer. [0.99c]

46. A particle moves with velocity represented by a vector $u' = 3\hat{i} + 4\hat{j} + 12\hat{k}\,ms^{-1}$ in S' frame. Find the velocity of the particle in frame S. S' moves with velocity 0.8 c relative to S along positive x-direction. $[u = 2.4\times10^{8}\hat{i} + 2.4\hat{j} + 7.2\hat{k}]$

47. S and S' are two inertial frames. S be rest and S' be moving with a uniform speed v. Find the coordinates of S' of the following.
    $v = 0.6c$
    a) $x = 4m$, $t = 1s$ [a) $x' = 2.5\times10^8 m$, $t' \simeq 1.25s$
    b) $x = 1.8\times10^8 m$, $t = 1s$ b) $x' = 0$, $t' = 0.8s$]

48. A rod of length l_0 oriented parallel to the x-axis moves with speed u along the x-axis in S. What is the length measured by an observer in S'. $\left[l = l_0\frac{[(c^2 - v^2)(c^2 - u^2)]^{\frac{1}{2}}}{c^2 - uv}\right]$

49. A photon of energy E_0 collides with a free particle of mass m_0 at rest. If the scattered photon flies at an angle θ, what is the scattering angle of the particle ϕ.
    $\left[\cot\phi = \left(1+\frac{E_0}{m_0c^2}\right)\tan\frac{\theta}{2}\right]$

50. An electron of energy 10 MeV moving at right angles to a uniform field 2T. Calculate the radius of the circular path a) classically (b) relativistically [(a) 0.53cm (b) 1.8 cm]

Section C

(*Answer questions in about* **one** *or* **two** *pages*)

Long answer type questions (Essays)

1. Describe the Michelson-Morley experiment and explain the null result obtained.
2. Derive the Lorentz transformation equations.
3. Explain the consequences of Lorentz transformation equations.

4. Derive velocity transformation equations from Lorentz transformation equations. How it is verified experimentally.
5. Derive the relativistic Doppler formula
6. How does the variation of mass with velocity has been verified.
7. By considering inelastic collision in relativity, establish that the inertial frames S and S′ are identical.
8. Explain the experimental verification of the equivalence of mass and energy.
9. Derive the Doppler formula on the basis of photon picture of light.
10. Explain how does light from a pulsar sets an upper limit to photon's rest mass.

Hints to problems

18. Number of expected fringe shift $=\dfrac{2Lv^2}{\lambda c^2}$

21. See example 2

22. $L=L_0\sqrt{1-v^2/c^2}$, $L=\frac{3}{4}L_0$ $\Delta t=\dfrac{\Delta t_0}{\sqrt{1-v^2/c^2}}$, $\Delta t=1$ second

One second in spacecraft will appear to be 0.75 second for an observer on earth.

23. $L_0=1\text{m}$ $\quad L=L_0\sqrt{1-v^2/c^2}$

24. See example 5

$L'=3\text{m}$ $\quad L'_x=L'\cos 60=\frac{3}{2}\text{m}$

$L'_y=L'\sin 60=\dfrac{3\sqrt{3}}{2}$ $\quad L_x=L'_x\sqrt{1-v^2/c^2}=0.9\text{m}$

$L_y=L'_y=\dfrac{3\sqrt{3}}{2}$

$\therefore \quad L=(L_x^2+L_y^2)^{\frac{1}{2}}=2.7495\text{m}$

25. $\Delta t_0=1\text{day}$

$\Delta t=2\text{days}$ $\quad$ use $\Delta t=\dfrac{\Delta t_0}{\sqrt{1-v^2/c^2}}$.

26. 60 minutes in rest clock appears to be 59 minutes in the moving clock. Then

$\Delta t_0=59$ minutes, $\Delta t=60$ minutes use $\Delta t=\dfrac{\Delta t_0}{\sqrt{1-v^2/c^2}}$.

27. $m=9.1\times10^{-31}\text{kg}$ use $E=mc^2$.

28. See example 14

29. $M_N\rightarrow M_p+M_e$

loss of mass $m = M_N - (M_p + M_e)$

This loss of mass is converted into energy use $E = mc^2$ in joules

To convert joule into MeV divide by 1.6×10^{-13}.]

30. $m_e = m_p = 9 \times 10^{-31}$ kg $\quad m_e c^2 + m_p c^2 = 2mc^2$

energy of each proton $= mc^2$

31. See example 17

32. $p = mv = m\dfrac{c}{\sqrt{2}}$, where $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}} = \sqrt{2}\, m_0$

$K = (m - m_0)c^2 \qquad T.E = K + m_0 c^2$

33. $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$ where $v = \dfrac{4}{5}c$

34. $\begin{bmatrix} E = mc^2, m = 1\text{kg} \\ 1\text{KWH} = 36 \times 10^5 \text{ joules.} \end{bmatrix}$

35. Let M be the initial mass.

Energy absorbed during melting $= M\,L$, $L = 336 \times 10^3\,\text{J kg}^{-1}$

This is equal to $mc^2 = 1 \times c^2$, $M\,L = mc^2$.

36. Show that $x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2$ using

$x' = \gamma(x - vt)$, $y' = y$, $z' = z$ and $t' = \gamma(t - vx/c^2)$

37. Use $E^2 = (p^2c^2 + m_0^2c^4)$ then replace p by mv, which is $\gamma m_0 v$.

38. $E = mc^2$, $m = m_0/\sqrt{1 - u^2/c^2}$, $E' = m'c^2$

$m' = m_0/\sqrt{1 - u'^2/c^2}$. Use $E' = \dfrac{E - vp_x}{\sqrt{1 - v^2/c^2}}$, $p_x = mu_x$

Substitute for E', E and p_x

39. Use $x' = \gamma(x - vt)$, $y' = y$, $z' = z$ and $t' = \gamma(t - vx/c^2)$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial}{\partial z'}\frac{\partial z'}{\partial x} + \frac{\partial}{\partial t'}\frac{\partial t'}{\partial x}$$

$$\frac{\partial}{\partial x'} = \gamma,\ \frac{\partial y'}{\partial x} = 0,\ \frac{\partial z'}{\partial x} = 0,\ \frac{\partial t'}{\partial x} = \frac{\gamma v}{c^2}$$

$$\therefore \quad \frac{\partial}{\partial x} = \gamma\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t}\right)$$

Similary $\frac{\partial}{\partial y}=\frac{\partial}{\partial y'}$ and $\frac{\partial}{\partial z}=\frac{\partial}{\partial z'}$, $\frac{\partial}{\partial t}=\gamma\left(\frac{\partial}{\partial t'}-\frac{v\partial}{\partial x'}\right)$

Find $\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}$ and $\frac{\partial^2}{\partial t^2}$. All these are put in given equation it will be

$$\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}.$$

40. Put $v^2=\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2$

41. $\Delta t_0 = 2.5\times10^{-8}$s, $\Delta t = 2.5\times10^{-7}$s

Using $\Delta t=\frac{\Delta t_0}{\sqrt{1-v^2/c^2}}$ find v.

42. $\Delta t_0 = 1.8\times10^{-8}$s, $v=0.6c$ find Δt using $\Delta t=\frac{\Delta t_0}{\sqrt{1-v^2/c^2}}$

43. For a circle $x^2+y^2=R^2$ use $x=\gamma(x'+vt')$ and $y=y'$

44. $u'_x=\frac{u_x-v}{1-\frac{u_xv}{c^2}}$ and $u'_y\frac{u_y\sqrt{1-v^2/c^2}}{1-\frac{u_xv}{c^2}}$

$u_x = u\cos\theta$ $\quad u_y = u\sin\theta$ and $\frac{u'_y}{u'_x}=\tan\theta'$

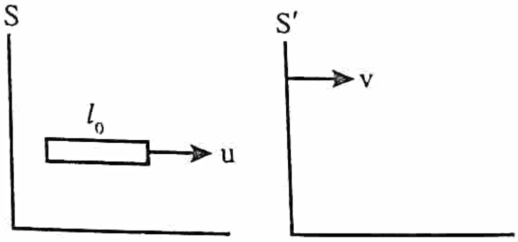
45. $u'_x=\frac{u_x-v}{1-\frac{u_xv}{c^2}}$, $u_x=.98c$, $v=0.98c$

46. $u'_x=3$ using $u_x=\frac{u'_x+v}{1+\frac{u'_xv}{c^2}}$ where $v=0.8c$

$u'_y=4$ using $u_y=\frac{u'_y\sqrt{1-v^2/c^2}}{1+\frac{u'_xv}{c^2}}$ $u'_z=12$ using $u_z=\frac{u'_z\sqrt{1-v^2/c^2}}{1+\frac{u'_xv}{c^2}}$

47. See example 9

48. Speed of the rod w.r.t. to S′, $V = \dfrac{u - v}{1 - \dfrac{uv}{c^2}}$

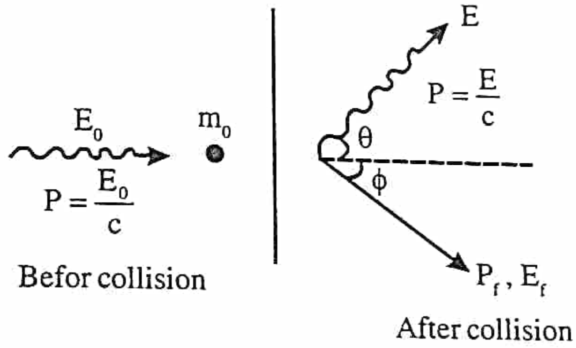


Using $L = L_0\sqrt{1 - v^2/c^2}$ .... (1)

Here $L = l'$, $L_0 = l$, $v = V = \dfrac{u - v}{1 - \dfrac{uv}{c^2}}$

substituting in eq (1), we get the result.

49.



From conservation of momentum along x-direction

$$\frac{E_0}{c} = \frac{E}{c}\cos\theta + p_f\cos\phi \qquad ..... (1)$$

along y-direction

$$0 = \frac{E}{c}\sin\theta + p_f\sin\phi \qquad ..... (2)$$

From conservation of energy, we have

$$E_0 + m_0c^2 = E + E_f \qquad ..... (3)$$

Where $E_f = \sqrt{p_f^2c^2 + m_0^2c^4}$ solving eqs (1), (2) and (3) collect ϕ.

50. a) $r = \dfrac{m_0 v}{qB} = \dfrac{p}{qB} = \sqrt{\dfrac{2m_0K}{qB}}$

$m_0 = 9.1\times10^{-31}$ kg, $K = 10$MeV $q = 1.6\times10^{-19}$ c, $B = 2T$ we get r = 0.53 cm

b) $r = \dfrac{mv}{qB} = \dfrac{p}{qB}$ ...... (1)

$E = \sqrt{p^2c^2 + m_0^2c^4}$ from this calculate p and put in eq(1), we get r=1.8cm

Where $E = m_0c^2 + K$

UNIT TWO

2 GENERAL RELATIVITY AND COSMOLOGY

Introduction

To study the behaviour of matter there are two fundamental theories in physics. They are

(i) Newtonian theory of gravitation - which describes the behaviour of one mass point (gravitational field) and

(ii) the electrodynamics - which describes the behaviour of charged matter in the presence of electromagnetic field.

The special theory of relativity had its origin in the development of electrodynamics, while the general theory of relativity is the relativistic theory of gravitation.

The special theory of relativity only accounts for inertial systems, in the regions of free space where gravitational effects are neglected. In those systems the law of inertia holds good and the physical laws retain the same form. The special theory of relativity does not account for non-inertial (accelerated) systems. For example the clock paradox and universal phenomenon of gravitation could not be accounted by special theory of relativity. Thus naturally we wish to extend the principle of relativity in such a way that it may hold even for non-inertial systems and consequently the extended theory may explain the non-inertial phenomenon like clock paradox and particularly the phenomenon of gravitation. This is because **when S.T.R is extended to accelerated systems we can easily bring the effect of gravitation. The extended theory is known as the general theory of relativity**. In developing the general theory of relativity it is helpful to analyse the predictions of special theory of relativity with respect to the phenomenon of gravitation.

Principle of equivalence is the keystone of general theory of relativity enunciation by Albert Einstein in 1915.

The principle of equivalence

This is actually the principle of equivalence of gravitation and inertia. **It states that there is no way to distinguish locally the motion produced by inertial forces (acceleration, recoil, centrifugal forces) from motion produced by gravitational force.**

Einstein arrived at this conclusion with the following gedanken (thought) experiment.

In accordance with his usual mode of creative thought, Einstein set the stage with an imaginary situation. He pictured an immensely high building and inside it an elevator that had slipped from its cables and is falling freely. Within the elevator a group of physicists, undisturbed by any suspicion that their ride might end in disaster, are performing experiments. They take objects from their pockets, a fountain pen, a coin, a bunch of keys, and release them from their grasp. Nothing happens. The pen, the coin, the keys appear to the men in the elevator to remain posed in mid-air-because all of them are falling, along with the elevator and the men, at precisely the same rate in accordance with Newton's law of gravitation. Since the men in the elevator are unaware of their predicament, however, they may explain these peculiar happenings by a different assumption. They may believe they have been magically transported outside the gravitational field of the earth and are in fact poised somewhere in empty space. And they have good grounds for such a belief. If one of them jumps from the floor he floats smoothly towards the ceiling with a velocity just proportional to the vigour of his jump. If he pushes his pen or his keys in any direction, they continue to move uniformly in that direction until they hit the wall of the car. Everything apparently obeys Newton's law of inertia, and continues in its state of rest or of uniform motion in a straight line. The elevator has somehow become an inertial system, and there is no way for the men inside it to tell whether they are falling in a gravitational field or are simply floating in empty space, free from all external forces.

Einstein now shifts the scene, The physicists are still in the elevator, but this time they really *are* in empty space, far away from the attractive power of any celestial body. A cable is attached to the roof of the elevator; some supernatural force begins reeling in the cable; and the elevator travels "upward" with constant acceleration, i.e., progressively faster and faster. Again the men in the car have no idea where they are, and again they perform experiments to evaluate their situation. This time they notice that their feet press solidly against the floor. If they jump they do not float to the ceiling, for the floor comes up beneath them. If they release objects from their hands, the objects appear to "fall". If they toss objects in a horizontal direction they do not move uniformly in a straight line, but describe a parabolic curve with respect to the floor. And so the scientists, who have no idea that their windowless car actually is climbing through interstellar space, conclude that they are situated in quite ordinary circumstances in a stationary room rigidly attached to the earth and affected in normal measure by the force of gravity. There is really no way for them to tell whether they are at rest in a gravitational field or ascending with constant acceleration through outer space where there is no gravity at all.

The same dilemma would confront them if their room were attached to the rim of a huge rotating merry-go-round set in outer space. They would feel a strange force trying to pull them away from the centre of the merry-go-round, and a sophisticated outside observer would quickly identify this force as inertia (or, as it is termed in the case of rotating objects, centrifugal force). But the men inside the room, who as usual are unaware of their odd predicament, would once again attribute the force to gravity. For if the interior of their room is empty and unadorned, there will be nothing to tell them which is the floor and which is the ceiling except the force that pulls them towards one of its interior surfaces. So what a detached observer would call the "outside wall" of the rotating room becomes the "floor" of the room for the men inside. A moment's reflection shows that there is no "up" or "down" in empty space. What we on earth call "down" is simply the direction of gravity. To a man on the sun it would appear that the Australians, Africans, and Argentines are hanging by their heels from the southern hemisphere. By, the same token, Admiral Byrd's flight over the South Pole was a geometrical fiction. actually he flew *under* it - upside down. And so the men inside the room on the merry-go-round will find that all their experiments produce exactly the same results as the ones they performed when their room was being swept "upward" through space. Their feet stay firmly on the "floor." Solid objects "fall". And once again they attribute these phenomena to the force of gravity and believe themselves at rest in a gravitational field.

From these fanciful occurances Einstein draw a conclusion of great theortical importance. To physicists it is known as the principle of equivalence of gravitation and inertia.

The above discussion shows that an accelerated frame can bring the effect of graitational force. For example an elevator moving in outer space moving upward with an acceleration a = g; bring gravitional effect.

In the statement of the principle of equivalence, we used the word "there is no way to distinguish locally.............., locally we mean that there is no way to distinguish from within a sufficiently confined system. Our elevator is such a system. In other words the equivalence principle applies only to local systems i.e., only for small systems inertial and gravitational forces are indistinguishable. For non-local systems they are distinguishable.

Einstein realised that the principle of equivalence applied not only to mechanical systems but to all experiments, even ones based on electromagnetic radiations. Principle of equivalence predicts a change in frequency of a light wave falling in the earth's gravity.

To illustrate this consider an elevator moving up with acceleration a. At the top of the elevator there is a light source that emits a wave of frequency υ. At the bottom of the elevator and a distance H away there is a detector that observes the wave and measures its frequency. When the light wave is emitted in the accelerating elevator, the source has speed v, which is assumed to small compared with the speed of light c. When the wave is detected by the detector after a time $t \simeq \frac{H}{c}$, the floor is moving with a speed v + at. In effect there is a relative speed $\Delta v = at$ between the source and the detector, so there is Doppler shift in frequency given by

$$\upsilon' = \upsilon\sqrt{\frac{1+\frac{\Delta v}{c}}{1-\frac{\Delta v}{c}}} = \upsilon\left(1+\frac{\Delta v}{c}\right)^{1/2}\left(1-\frac{\Delta v}{c}\right)^{1/2}$$

$$\upsilon' \simeq \upsilon\left(1+\frac{\Delta v}{2c}\right)\left(1+\frac{\Delta v}{2c}\right)$$

$$\upsilon' \simeq \upsilon\left[1+\frac{\Delta v}{2c}+\frac{\Delta v}{2c}+\left(\frac{\Delta v}{2c}\right)^2\right]$$

Neglecting $\left(\frac{\Delta v}{2c}\right)^2$ we get

$$\upsilon' \simeq \upsilon\left(1+\frac{\Delta v}{c}\right)$$

$\therefore$

$$\frac{\upsilon'-\upsilon}{\upsilon} = \frac{\Delta v}{c}$$

$$\frac{\upsilon'-\upsilon}{\upsilon} = \frac{at}{c} \qquad (\because \quad \Delta v = at)$$

If the acceleration of the elevator a = g

$$\frac{\Delta\upsilon}{\upsilon} = \frac{gt}{c}$$

$$\frac{\Delta\upsilon}{\upsilon} = \frac{gH}{c^2} \qquad \left(\because \quad t = \frac{H}{c}\right)$$

Thus we can say that the principle of equivalence predicts a change in frequency of a light wave falling in the earth's gravity. This has been verified experimentally by Pound and Rebka in 1959. Further the frequency of radiation emitted by satellites and received by ground stations have confirmed this prediction to a precision of about 1 part in 10^4. This is made use of in the global positioning system (GPS). GPS relies on frequency measurements on the surface of the earth from transmitters in orbiting satellites, its accuracy depends on applying correction due to the gravitational frequency shift predicted by general relativity. Without this correction, errors in the GPS locating system of roughly 10 km per day would accumulate.

The equation for frequency shift can be re-written as

$$\frac{\Delta\upsilon}{\upsilon} = \frac{mgH}{mc^2}$$

$\frac{mgH}{m}$ is the difference in gravitational potential energy per unit mass (ΔV) between the source and the detector.

Thus $$\frac{\Delta\upsilon}{\upsilon} = \frac{\Delta V}{c^2}$$

suppose light leaving the surface of a star of mass M and radius R. The gravitational potential at the surface is $V = -\frac{GM}{R}$. If the light is observed on the earth, where the gravitational potential is negligible compared with that of the star, the frequency shift is

$$\frac{\Delta\upsilon}{\upsilon} = \frac{\Delta V}{c^2} = -\frac{GM}{Rc^2}$$

photons climbing out of a star's gravitational field lose energy and therefore shifted to smaller frequencies or longer wavelengths.

i.e. the photon has a lower frequency at the earth, corresponding to its less in energy *as it leaves the field of the star*. A photon in the visible region of the spectrum is thus *shifted towards the red end* and this phenomenon is known as gravitational red shift.

Example 1

The Lyman α line in the hydrogen spectrum has a wavelength of 121.5mm. Find the change in wavelength of this line in the solar spectrum due to the gravitational field. $G = 6.67\times10^{-11}\,Nm^2kg^{-2}$

$$M_\odot = 1.99\times10^{30}\,kg \text{ and } R_\odot = 6.96\times10^{8}\,m$$

Solution

We have $\dfrac{\Delta\upsilon}{\upsilon} = -\dfrac{GM}{Rc^2}$

Using $\upsilon = \dfrac{c}{\lambda}$

$$\Delta\upsilon = -\frac{c}{\lambda^2}\Delta\lambda$$

$$\therefore \quad \frac{\Delta\upsilon}{\upsilon} = -\frac{\Delta\lambda}{\lambda}$$

or
$$\frac{\Delta\lambda}{\lambda} = -\frac{\Delta\upsilon}{\upsilon} = \frac{GM}{Rc^2}$$

substituting the values, we get

$$\frac{\Delta\lambda}{121.5\times10^{-9}} = -\frac{6.67\times10^{-11}\times1.99\times10^{30}}{6.96\times10^{8}\times(3\times10^{8})^2}$$

$$\frac{\Delta\lambda}{121.5\times10^{-9}} = 2.119\times10^{-6}$$

or
$$\Delta\lambda = 2.119\times10^{-6}\times121.5\times10^{-9}$$

$$\Delta\lambda = 0.257\,pm$$

Example 2

The increase in energy of a fallen photon was first observed in 1960 by Pound and Rebka at Harvard. Find the change in frequency of a photon of red light whose original frequency is $7.3\times10^{14}\,Hz$ when it falls through 22.5 m

solution

We have $\frac{\Delta \upsilon}{\upsilon} = \frac{gH}{c^2}$

Substituting the values, we get

$$\frac{\Delta \upsilon}{7.3 \times 10^{14}} = \frac{9.8 \times 22.5}{(3 \times 10^8)^2}$$

or $$\Delta \upsilon = \frac{9.8 \times 22.5}{9 \times 10^{16}} \times 7.3 \times 10^{14}$$

$$\Delta \upsilon = 1.79 \text{Hz}$$

General theory of relativity

In Newtonian mechanics space and time were kept apart whereas in relativity space and time are intimately coupled via Lorentz transformation and called them as space time. General relativity is a theory of geometry. The motion of a particle is determined by the properties of space and time coordinates through which it moves. The equivalence between accelerated motion and gravity suggests a relationship between space time coordinates and gravity. In classical description, we would say that the presence of matter sets up a gravitational field, which then determines how objects move in response to that field. **According to general relativity the presence of matter causes space time to warp or curve, the motion of particles is determined by the shape of the curvature of space time.** General relativity gives us a procedure for calculating the curvature of space time. Roughly this is given by

$$\textbf{Curvature of space} = \frac{8\pi G}{c^4} \textbf{ energy momentum}$$

where G is the universal gravitational constant. This equation is called Einstein's field equation in general relativity. When there is no mass, energy momentum is zero so the curvature is zero and space is flat. In the limiting case $c \rightarrow \infty$ (classical mechanics) and in the limit of weak gravitational field the Einstein's field equation becomes Newton's law of gravitation.

In non-technical terms what is actually Einstein's theory of gravitation. The gravitation of Einstein is entirely different from the gravitation of Newton. It is not a force. Einstein's law of gravitation contains nothing about force. It describes the behaviour of objects in a gravitational field - the planets, for example - not in terms of attraction but simply in terms of the paths they follow.

To Einstein, gravitation is simply a part of inertia; the movement of stars and the planets stem from their inherent inertia, and the courses they follow are determined by the metric properties of space. Although this sounds very abstract and even paradoxical, it becomes quite clear as soon as one dismisses the notion that bodies of matter can exert physical force $\left(F = \frac{GMm}{r^2}\right)$ on each other across millions of kilometres of empty space. This concept of "action at a distance" has troubled scientists since Newton's day. It led to particular difficulty in understanding electric and magnetic phenomena. Today scientists no longer say that a magnet "attracts" a piece of iron by some kind of mysterious but instantaneous action-at-a distance. They say rather that the magnet creates a certain physical condition in the space around it, which they term a magnetic field; and that this magnetic field then acts upon the iron and makes it behave in a certain predictable fashion. Students in any elementary science course know what a magnetic field looks like, because it can be rendered visible by the simple process of shaking iron filings on to a piece of stiff paper held above a magnet. A magnetic field and an electrical field are physical realities. They have a definite structure, and their structure is described by the field equations of James Clerk Maxwell which pointed the way towards all the discoveries in electrical and radio engineering of the past century. A gravitational field is as much of a physical reality as an electromagnetic field, and its structure is defined by the field equations of Albert Einstein.

Just as Maxwell and Faraday assumed that a magnet creates certain properties in surrounding space, so Einstein concluded that stats, moons, and other celestial objects individually determine the properties of the space around them. And just as the movement of a piece of iron in a magnetic field is guided by the structure of the field, so the path of any body in a gravitational field is determined by the geometry of that field. The distinction between Newton's and Einstein's ideas about gravitation has sometimes been illustrated by picturing a little boy playing marbles in a city lot. The ground is very uneven, ridged with bumps and hollows. An observer in an office ten stories above the street would not be able to see these irregularities in the ground. Noticing that the marbles appear to avoid some sections of the ground and move towards other sections, he might assume that a "force" is operating which repels the marbles from certain spots and attracts them towards others. But another observer on the ground would instantly perceive that the path of the marbles is simply governed by the curvature of the field. In this little fable Newton is the upstairs observer who imagines that a "force" is at work, and Einstein is the observer on the ground, who has no reason to make such an assumption. Einstein's gravitational laws, therefore, merely describe the field properties of the space-time con-tinuum. Specifically, one group of these laws sets forth the relation between the mass of a gravitating body and the structure of the field around it; they are called structure laws. A second group analyses the paths described by moving bodies in gravitational fields; they are the laws of motion.

Tests of general relativity

It should not be thought that Einstein's theory of gravitation is only a formal mathematical scheme. For it rests on assumptions of deep cosmic significance. And the most remarkable of these assumptions is that the universe is not a rigid and immutable edifice where independent matter is housed in independent space and time; it is on the contrary an amorphous continuum, without any fixed architecture, plastic and variable, constantly subject to change and distortion. Wherever there is matter and motion, the continuum is disturbed. Just as a fish swimming, in the sea agitates the water around it, so a star, a comet, or a galaxy distorts the geometry of the space-time through which it moves.

When applied to astronomical problems, Einstein's gravitational laws yield results that are close to those given by Newton. If the results paralleled each other in every case, scientists might tend to retain the familiar concepts of Newtonian law and write off Einstein's theory as a weird if original fancy. But a number of strange new phenomena have been discovered, and at least one old puzzle solved, solely on the basis of general relativity. The old puzzle stemmed from the eccentric behaviour of the planet mercury. Instead of revolving in its elliptical orbit with the regularity of the other planets, mercury deviates from its course each year by a slight but exasperating degree. Astronomers explored every possible factor that might cause this perturbation, but found no solution within the framework of Newtonian theory. It was not until Einstein evolved his laws of gravitation that the problem was solved. Of all the planets mercury lies closest to the sun. It is small and travels with great speed. Under Newtonian law these factors

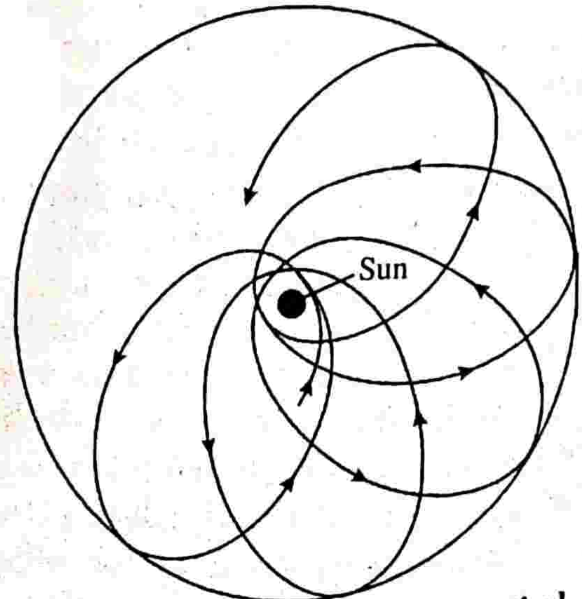


Figure 2.1: The rotation of mercury's elliptical orbit, greatly exaggerated. Actually the ellipse advances only 43 seconds of an arc per century.

should not in themselves account for the deviation; the dynamics of Mercury's movement should be basically the same as those of any other planet. But under Einstein's laws, the intensity of the sun's gravitational field and mercury's enormous speed make a difference, causing the whole ellipse of mercury's orbit to execute a slow but inexorable swing around the sun at the rate of one revolution in 3,000,000 years. This calculation is in perfect agreement with actual measurements of the planet's course. Einstein's mathematics are thus more accurate than Newton's in dealing with high velocities and strong gravitational fields.

Another prediction made by Einstein was the effect of gravitation on light.

The Sequence of thought which led Einstein to prophesy this effect began with another imaginary situation. As before, the scene opens in an elevator ascending with constant acceleration through empty space, far from any gravitational field. This time some roving interstellar gunman impulsively fires a bullet at the elevator. The bullet hits the side of the car, passes clean through and emerges from the far wall at a point a little below the point at which it penetrated the first wall. The reason for this is evident to the marksman on the outside. He knows that the bullet flew in a straight line, obeying Newton's law of Inertia; but while it traversed the distance between the two walls of the car, the whole elevator travelled "upward" a certain distance, causing the second bullet hole to appear not opposite the first one but slightly nearer the floor. However, the observers inside the elevator, having no idea where in the universe they are, interpret the situation differently. Aware that on earth any missile describes a parabolic curve towards the ground, they simply conclude that they are at rest in a gravitational field and that the bullet which passed through their car was describing a perfectly normal curve with respect to the floor.

A moment later as the car continues upwards through space a beam of light is suddenly flashed through an aperture in the side of the car. Since the velocity of light is great, the beam traverses the distance between its point of entrance and the opposite wall in a very small fraction of a second. Neverthe-less, the car travels upwards a certain distance in that interval, so the beam strikes the far wall a tiny fraction of an inch below the point at which it entered. If the observers within the car are equipped with sufficiently delicate instruments of measurement, they will be able to compute the curvature of the beam. But the question is, how will they explain it? They are still unaware of the motion of their car and believe themselves at rest in a gravitational field. If they cling to Newtonian principles, they will be completely baffled because they will insist that light rays always travel in a straight line. But if they are familiar with the special theory of relativity they will remember that energy has mass in accordance with the equation $m = E/c^2$. Since light is a form of energy they will deduce that light has mass and will therefore be affected by a gravitational field. Hence the curvature of the beam.

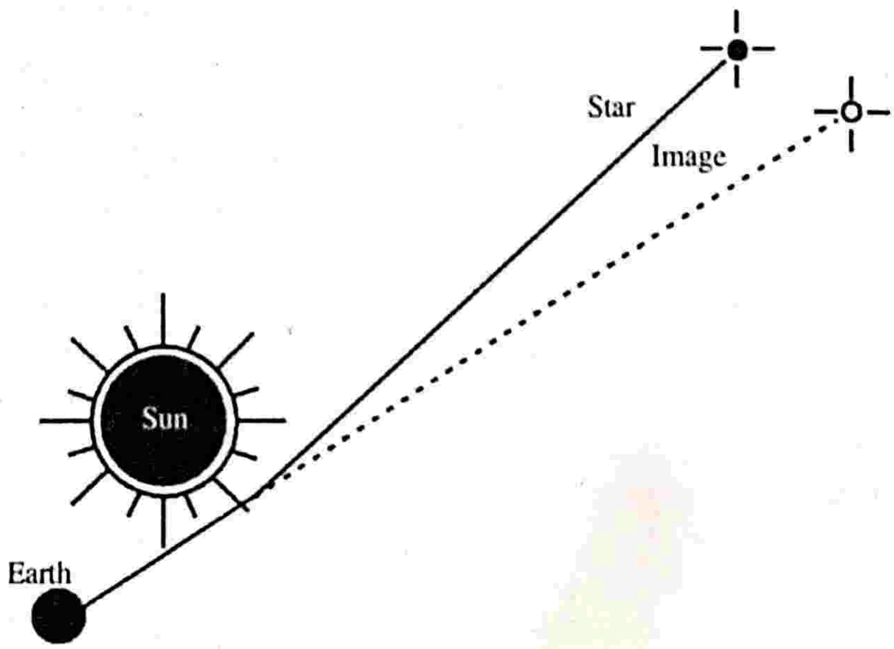


Figure 2.2: The deflection of starlight in the gravitational field of the sun. Since the light from a star in the neighbourhood of the sun's disk is bent in wards, towards the sun, as it passes through the sun's gravitational field, the image of the star appears to observers on earth to be shifted outwards and away from the sun.

From these purely theoretical considerations Einstein concluded that light, like any material object, travels in a curve when passing through the gravitational field of a massive body. He suggested that his theory could be put to test by observing the path of starlight in the gravitational field of the sun. Since the stars are invisible by day, there is only one occasion when sun and stars can be seen together in the sky, and that is during an eclipse. Einstein proposed, therefore, that photographs be taken of the stars immediately bordering the darkened face of the sun during an eclipse and compared with photographs of those same stars made at another time.

According to his theory, the light from the stars surrounding the sun should be bent inwards, towards the sun, in traversing the sun's gravitational field; hence the *images* of those stars should appear to observers on earth to be shifted outwards from their usual positions in the sky. Einstein calculated the degree of deflection that should be observed and predicted that for the stars closest to the sun the deviation would be about 1.75 seconds of an arc. Since he staked his whole general theory of relativity on this test, men of science throughout the world anxiously awaited the findings of expeditions which journeyed to equatorial regions to photograph the eclipse of 29 May 1919. When their pictures were developed and examined, the deflection of the starlight in the gravitational field of the sun was found to average

1.64 seconds — a figure as close to perfect agreement with Einstein's prediction as the accuracy of instruments allowed.

Another prediction made by Einstein on the basis of general relativity pertained to time. Having shown how the properties of space are affected by a gravitational field, Einstein reached the conclusion by analogous but somewhat more involved reasoning that time intervals also vary with the gravitational field. A clock transported to the sun should run at a slightly slower rhythm than on earth. And a radiating solar atom should emit light of slightly lower frequency than an atom of the same element on earth. The difference in wavelength would in this case be immeasurably small. But there are in the universe gravitational fields stronger than the sun's. One of these surrounds the freak star known as the "companion of Sirius" — a white dwarf composed of matter in a state of such fantastic density that 1 cubic inch of it would weigh a ton on earth. Because of its great mass, this extraordinary dwarf, which is only three times larger than the earth, has a gravitational field potent enough to perturb the movements of Sirius, seventy times its size. Its field is also powerful enough to slow down the frequency of its own radiation by a measurable degree, and spectroscopic observations have indeed proved that the frequency of light emitted by Sirius' companion is reduced by the exact amount predicted by Einstein. The shift of wavelength in the spectrum of this star is known to astronomers as "the Einstein Effect" and constitutes an additional verification of general relativity.

Stellar evolution

The predictions of general relativity were experimentally verified by the presence of large gravitational field produced by the sun. Another remarkable experimental verification of general relativity is due to much more gravitational field produced by the collapse of stars. On the basis of general relativity we predicted the stellar evolution and their collapses. A star collapsed into more compact objects like white dwarfs, neutron stars and blackholes etc. in their final stage. We predicted these and experimentally detected them provided addition verification for the presence of gravitational field and the general theory of relativity. Stellar evolution will be discussed extensively in the forth coming chapter.

The expansion of the universe

The picture of the origin of the universe began with the formulation of general theory of relativity in 1915 by Albert Einstein. Solving the field equations Einstein got an expanding universe. But he believed that the universe is static. So he modified his theory to make this possible by introducing a so called cosmological constant into his equations (one of the greatest blunders that Einstein ever committed, he himself admitted this). That is Einstein introduced a new antigravity force unlike other forces did not come from any particular source but was built into the very fabric of space time. He claimed that space -time had an inbuilt tendency to expand and this could be made to balance exactly the attraction of all the matter in the universe, so that a static universe would result. Einstein and other physicists were looking for ways of avoiding general relativities prediction of a non-static universe but one man the Russian physicist and mathematician Alexander Friedmann instead set about explaining it.

Friedmann made two very simple assumptions about the universe, one is that the universe looks identical in which ever direction we look, the second one is that this would also be true if we were observing the universe from anywhere else. From these two assumptions in 1922 Friedmann showed that our universe is non-static. Few years later (1929) the American astronomer Edwin Hubble experimentally proved that we are living in an expanding universe. The discovery that the universe is expanding was one of the greatest intellectual revolutions of the twentieth century.

The evidence for the expansion of the universe comes from the observed change in the wavelength of the light emitted by distant galaxies. According to relativistic Doppler effect in terms of wavelength, we can write

$$\lambda' = \lambda\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

where λ is the wavelength emitted by the galaxy in its own frame, λ' is the wavelength we measure on earth and v is the relative velocity between the source of light and the observer.

The light emitted by a star such as the sun has a continuous spectrum. As the light passes through the stars atmosphere some of it is absorbed by the gases in the atmosphere, so the continuous emission spectrum has a few dark absorption lines super imposed. Comparison between the known wavelengths of these lines (measured on earth for sources at rest relative to the observer) and the Doppler shifted wavelengths enables us to calculate the speed of the star from the above equation.

If stars are moving away from us wavelength observed is found to be increased. i.e, wavelengths shifted towards the longer wavelength (red) and stars moving towards us wavelength observed is found to be shifted towards shorter wavelength (blue). The average speed of these stars in our galaxy relative to earth is about $3\times10^3 ms^{-1}$. The change in wavelength for these stars is very small. Light from nearby galaxies also show small change in red shift or blue shift.

However when we observe light from distant galaxies all are found to be red shifted. From these we concluded that galaxies are receding from us. According to Friedmann's assumption we can conclude that any other observer in the universe would draw the same conclusion. Combining the experimental observation and the assumption of Friedmann, in general we can say that galaxies would be observed to recede from every point in the universe. In other words our universe is expanding.

Hubble's law

In 1920 the American astronomer Hubble Edwin Powel (1889-1953) started using the 100 inch (1.5m) telescope on Mount Wilson in California. From this observation he concluded that spiral nebulae are not nebulae but distant galaxies. By measuring the red shift of these distant galaxies he calculated the speeds of these galaxies. At the same time he also devised a method to calculate the distances of these galaxies. Hubble observed 46 galaxies and their distances and speeds. From this he made two remarkable conclusions. The galaxies are moving away from us and the further away a galaxy is, the faster it is moving away. **This proportionality between the recessional speed (v) of galaxies and its distance is known as Hubble law.**

i.e., $v \propto d$

$$v = H_0 d$$

The proportionality constant H_0 is known as the Hubble parameter.

The Hubbles plot of v versus d is shown in figure below. It is due to the uncertainties in the measurement of the distance points obtained are found to be scattered. However it gives a strong indication that v and d are linear.

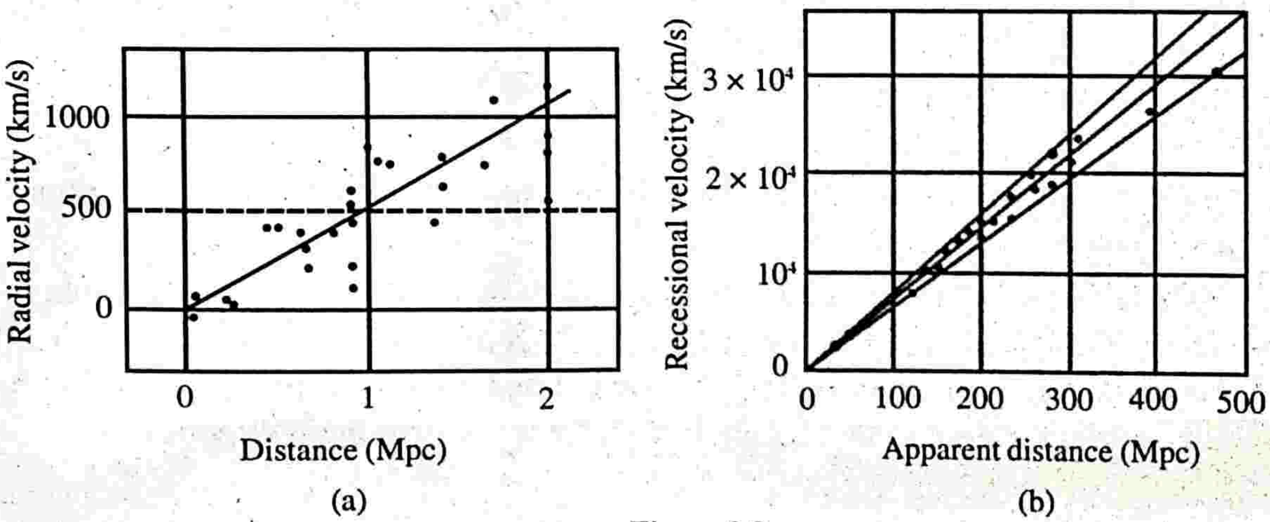


Figure 2.3

More modern data based on observing supernovas in distant galaxies are shown in figure above. From this the value of H_0 was calculated. The slope v-d graph gives H_0. The value of Hubble parameter calculated to be

$$H_0 = 72 kms^{-1}(Mpc)^{-1}$$

$$1 Mpc = \text{Million parsec}$$

$$1 \text{ Parsec} = 3.08 \times 10^{16} m$$

$$\therefore \quad H_0 = \frac{72 \times 10^3 ms^{-1}}{3.08 \times 10^{22} m} = 23.38 \times 10^{-19} s^{-1}.$$

The Hubble parameter has the dimension of inverse time.

The inverse of H_0 gives

$$\frac{1}{H_0} = \frac{1}{23.38 \times 10^{-19}} s$$

$$\frac{1}{H_0} = \frac{100 \times 10^{17}}{23.38} s = 4.28 \times 10^{17} s$$

or $$\frac{1}{H_0} = 4.28 \times 10^{17} s$$

or $$\frac{1}{H_0} = \frac{4.28 \times 10^{17}}{3.15 \times 10^7} \text{ years}$$

$$\frac{1}{H_0} = 13.59 \times 10^9 \text{ years}$$

Very surprisingly you can see that the inverse of Hubble parameter is almost equal to age of the universe.

Hubble's law was enunciated in the year 1929. After about 20 years Hubble installed a 200 inch (3m) Palomer telescope. Through this telescope he observed distant galaxies in all directions and concluded that what Friedmann said was perfectly correct, it follows that distribution matter is uniform in all directions.

Note: $1\ \text{AU} = 1.496 \times 10^{11}\ \text{m}$

$1\ ly = 9.46 \times 10^{15}\ \text{m}$

$1\ \text{pc} = 3.26 ly = 3.08 \times 10^{16}\ \text{m}$

Relation between v and d

Consider the universe represented by the three dimensional coordinate system shown in figure below, where each point is a galaxy, with the earth at the origin we can determine the distance of each galaxy. Consider two galaxies I and II. Let d_1 be the distance of galaxy I and d_2 that of galaxy II from the origin. If this universe were to expand with all points becoming farther apart. Then d_1 goes to d_1' and d_2 goes to d_2' (see fig. below). So we can write

$$d_1' = k\ d_1$$

$$d_2' = k\ d_2$$

or in general we can write

$$d' = k\ d$$

$\therefore$ The recessional velocity of any galaxy is

$$v = \frac{d' - d}{t}$$

where t is the time in which the distance d goes to d'.

or $$v = \frac{kd - d}{t} = \frac{d(k-1)}{t}$$

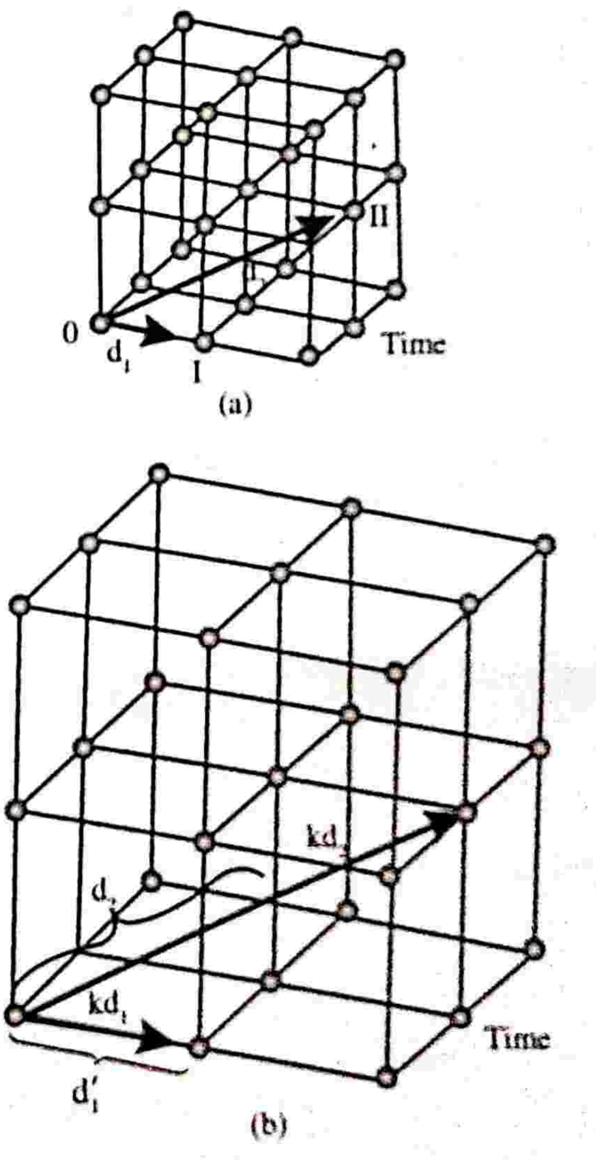


Figure 2.4: The expansion of a coordinate space, showing that the apparent speed of recession depends on the distance, d_2 is greater than d_1, and d_2 increases faster than d_1

If we compare two galaxies, we have

$$\frac{v_1}{v_2} = \frac{d_1}{d_2}$$

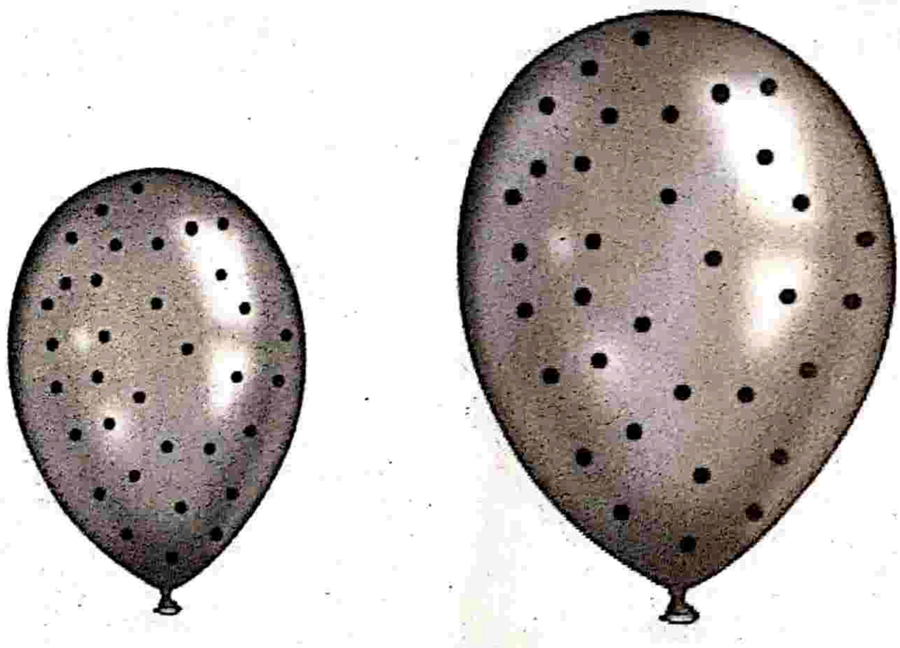
This is identical with Hubble's law

$$v = H_0 d$$

i.e. $v \alpha d$

This shows that if distance of galaxy is large recessional velocity of the galaxy is also large.

We can demonstrate the expanding universe with an analogy. Take a spotted balloon. Each spot represents a galaxy. When we inflate the balloon, one can see that each spot is moving away from the others. (see figure below)



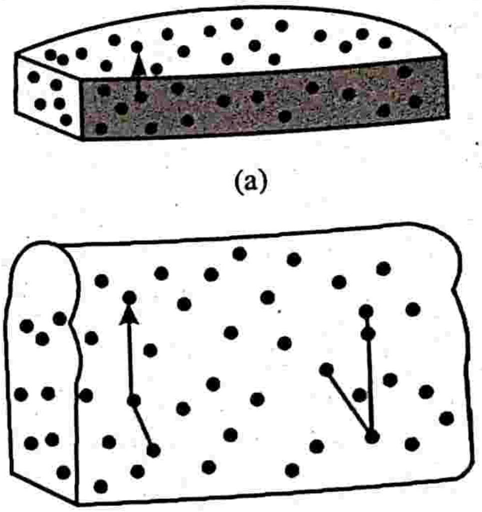
(a) (b)

Figure 2.5: As a balloon is inflated, every observer on the surface experiences a velocity distance relationship of the form of the Hubble law

Consider another analogy. Take a loaf of raisin bread and place it in an oven. As the bread expands every raisin observes all the others to be moving away from it and the speed of recession increases with the separation. See figure below.

So far we arrived at the expansion of the universe on the basis of relativistic

Doppler effect. Actually correct interpretation of the cosmological red shift should come from general theory of relativity. According to general relativity the shift in wavelength is caused by a stretching of the entire fabric of space-time. Imagine small photos of galaxies glued to a rubber sheet. As the sheet is stretched, the distance between the galaxies increases. This stretching of the space between the galaxies causes the wavelength of light signal from one galaxy to increase. At low speeds Doppler red shift gives correct results. However for very large cosmological red shifts, a more correct analysis must be based on the stretching model. According to stretching model



(a)

(b)

Figure 2.6: Another system in which the Habble law is valid

$$\frac{\lambda'}{\lambda} = \frac{R_0}{R}$$

where R_0 represents size of the universe at present and R represents the size of the universe at the time light was emitted. λ' is the wavelength of light received now and λ is the wavelength light when it was emitted.

Example 2

A distant galaxy in the constellation Hydra is receding from the earth at $6.12 \times 10^7 \, ms^{-1}$. By how much is a green spectral line of wavelength 500nm emitted by the galaxy shifted towards the red end of the spectrum. Also calculate the galaxy distance. Take $\frac{1}{H_0} = 14 \times 10^9$ years.

Solution

$$\lambda = 500 \times 10^{-9} m$$

$$v = 6.12 \times 10^7 ms^{-1}$$

$$\therefore \quad \frac{v}{c} = \frac{6.12 \times 10^7}{3 \times 10^8} = 0.204$$

Using $\lambda' = \lambda\sqrt{\dfrac{1+\frac{v}{c}}{1-\frac{v}{c}}} = 500\times10^{-9}\sqrt{\dfrac{1+0.204}{1-0.204}}$

$$\lambda' = 500\times10^{-9}\sqrt{\frac{1.204}{0.796}} = 500\times10^{-9}\times1.23$$

$$\lambda' = 615\text{nm}$$

$\therefore$ The shift in wavelength $=615-500\text{nm} = 115\text{nm}$

Using Hubble's law

$$v = H_0 d$$

$$d = \frac{v}{H_0} = 0.204c\times14\times10^9 \text{ years}$$

$$d = 0.204\times14\times10^9 \text{ light years}$$

$$d = 2.856\times10^9 \text{ light years}$$

Example 3

Use Hubble law to estimate the wavelength of the 590nm sodium line as observed emitted from galaxies whose distance from us is (a) 1.0×10^6 light years (b) 1.0×10^9 light years. Take $H_0 = 72\text{kms}^{-1}(\text{Mpc})^{-1}$

Solution

$$\lambda = 590\times10^{-9}\text{m}$$

a) $d = 1.0\times10^6$ light years

Using $v = H_0 d$ ..... (1)

$H_0 = 72\text{kms}^{-1}(\text{Mpc})^{-1}$

1 parsec = 3.26 ly

$\therefore$ $d = \dfrac{1.0\times10^6}{3.26}$ parsec $= 0.307\times10^6$ parsec

$d = 0.307\text{Mpc}$

Put this in equation 1, we get

$$v = 72\frac{km}{s}\cdot\frac{1}{Mpc}\cdot 0.307Mpc$$

$$v = 22.1\frac{km}{s} = 22.1\times10^{3}ms^{-1}$$

$$\therefore \quad \frac{v}{c} = \frac{22.1\times10^{3}}{3\times10^{8}} = 7.37\times10^{-5}$$

Using $\lambda' = \lambda\sqrt{\dfrac{1+\frac{v}{c}}{1-\frac{v}{c}}} = 590\times10^{-9}\sqrt{\dfrac{1+7.37\times10^{-5}}{1-7.37\times10^{-5}}}$

$$\therefore \quad \lambda' \simeq 590\times10^{-9} = 590nm$$

b) $d = 1.0\times10^{9}ly = \dfrac{1.0\times10^{9}}{3.26}pc$

$$d = 3.07\times10^{8}pc = 307Mpc$$

$$v = H_0 d$$

$$v = 72\frac{km}{s\ Mpc}\cdot 307Mpc$$

$$v = 22104\frac{km}{s} = 2.21\times10^{7}ms^{-1}$$

$$\frac{v}{c} = \frac{2.21\times10^{7}}{3\times10^{8}} = 7.37\times10^{-2}$$

$$\therefore \quad \lambda' = \lambda\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} = 590\times10^{-9}\sqrt{\frac{1+7.37\times10^{-2}}{1-7.37\times10^{-2}}}$$

$$\lambda' = 590\times10^{-9}\sqrt{\frac{1.0737}{0.9263}}$$

$$\lambda' = 590 \times 10^{-9} \times 1.08$$

$$\lambda' = 637.2 \times 10^{-9} \text{m}$$

$$\lambda' = 637.2 \text{nm}$$

The cosmic microwave background radiation

In 1923 Friedmann showed that our universe is non-static. Few years later (1929) Hubble experimentally confirmed the expansion of the universe. In 1940 George Gamow, a student of Freidmann, suggested that if we are living in an expanding universe and when we go back in time we get the early universe which must be hot and dense. Far enough back in time, the universe would have been too hot for stable matter to form. Its composition was then a gas of particles and photons.

The unstable particle eventually decayed to stable ones and the stable particles eventually clumped together to form matter. The photons that filled the universe remained but their wavelengths were stretched by the continuing expansion of the universe. Today those photons have a much lower temperature but they still uniformly fill the universe. This was first suggested by two American scientists. Bob Dickse and Jim Peebles. They argued that if Gamow said is correct we should still be able to see the glow of the early universe, because light form very distant parts of it would only be reaching us now. This **glow of the early universe is known as the cosmic back ground radiation.**

This picture of a hot early stage of the universe was first put forward by the scientist George Gamow in a famous paper written in 1948 with his student Ralph Alpher. Gamow had quite a sense of humour- he persuaded the nuclear scientist Hans Bethe to add his name to the paper to make the list of authors "Alpher, Bethe, Gamow'' like the first three letters of the Greek alphabet. Alpha, beta, gamma. Particularly appropriate for a paper on the beginning of the universe! In this paper they made the remarkable prediction that radiation (in the form of photons) from the very hot early stages of the universe should still be around today, but with its temperature reduced to only a few degrees above absolute zero(-273°C). At the time of publishing this paper, not much was known about the nuclear reactions of protons and neutrons. So predictions made for the proportions of various elements in the early universe were rather inaccurate. Gamow and others predicted that the background radiation would be at a temperature of the order of 5K to 10K and energy of the order of 10^{-3} eV or a wavelength of order 1mm. As this wavelength lies in the microwave region. The background radiation is called microwave back ground radiation. So they could not predict the temperature or energy of the back ground radiation correctly. But these calculations have been repeated in the light of better knowledge and now agree very well what we observe.

Calculation of the energy of the microwave background radiation

Consider the whole universe as a black body emitting radiation at the temperature T. From this we can calculate the average energy per photon at temperature T.

According to Planck's radiation law, the number of photons in the frequency range υ and $\upsilon + d\upsilon$ is

$$n(\upsilon)d\upsilon = \frac{8\pi\upsilon^2 V d\upsilon}{c^3} \frac{1}{e^{\frac{h\upsilon}{kT}} - 1} \qquad \text{..... (1)}$$

Since each photon has energy $h\upsilon$, the energy density

$$u(\upsilon) = \frac{\text{Total energy}}{\text{Volume}} = \frac{n(\upsilon)h\upsilon}{V}$$

$$\therefore \qquad u(\upsilon)d\upsilon = \frac{8\pi h}{c^3} \frac{\upsilon^3 d\upsilon}{e^{\frac{h\upsilon}{kT}} - 1} \qquad \text{..... (2)}$$

To get the total energy density we have to integrate the above equation over all possible frequencies i.e, from $\upsilon = 0$ to $\upsilon = \infty$

Thus $$u = \int_0^\infty u(\upsilon)d\upsilon = \frac{8\pi h}{c^3} \int_0^\infty \frac{\upsilon^3 d\upsilon}{e^{\frac{h\upsilon}{kT}} - 1}$$

Put $$\frac{h\upsilon}{kT} = x, \quad \text{then} \quad \frac{hd\upsilon}{kT} = dx$$

$$\therefore \qquad u = \frac{8\pi h}{c^3}\left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3}{e^x - 1} dx$$

The value of $$\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}$$

$$u = \frac{8\pi^5 k^4 T^4}{15 c^3 h^3} \qquad \text{..... (3)}$$

Integrating equations (1) from $\upsilon = 0$ to $\upsilon = \infty$ we get the total number of photons.

$$N = \int_0^\infty n(\upsilon)d\upsilon = \frac{8\pi V}{c^3}\int_0^\infty \frac{\upsilon^2 d\upsilon}{e^{\frac{h\upsilon}{kT}} - 1}$$

Put $\frac{h\upsilon}{kT} = x$, then $\frac{hd\upsilon}{kT} = dx$, we get

$$N = \frac{8\pi}{c^3}\left(\frac{kT}{h}\right)^3 V \int_0^\infty \frac{x^2 dx}{e^x - 1}$$

Then value of $\int_0^\infty \frac{x^2 dx}{e^x - 1} = 2.404$ (See example 4)

$$\therefore \quad \frac{N}{V} = \frac{8\pi k^3 T^3}{c^3 h^3} \cdot 2.404$$

$$\frac{N}{V} = \frac{19.23\pi k^3 T^3}{c^3 h^3} \qquad .....(4)$$

$\therefore$ The average energy of the photon

$$\bar{E} = \frac{\text{Total energy}}{\text{Total number}} = \frac{U}{N}$$

or

$$\bar{E} = \frac{U}{V\frac{N}{V}} = \frac{u}{\frac{N}{V}}$$

$$\bar{E} = \frac{8\pi^5 k^4 T^4}{15c^3 h^3} \cdot \frac{c^3 h^3}{19.23\pi k^3 T^3}$$

$$\bar{E} = \frac{8}{15 \times 19.23}\pi^4 kT$$

Substituting the value of $k = 0.8625 \times 10^{-4}\, eVK^{-1}$

we get $$\bar{E} = \frac{8 \times \pi^4 \times 0.8625 \times 10^{-4} T}{15 \times 19.23} eV.$$

$$\bar{E} = 2.325 \times 10^{-4} T \text{ eV}.$$

This is the expression for average energy per photon in terms of temperature T.

Now we look at the experimental evidence for the existence of microwave background radiation and the determination of its temperature.

Put $\upsilon = \frac{c}{\lambda}$, then $d\upsilon = -\frac{c}{\lambda^2} d\lambda$ in equation (2), we get

$$u(\lambda)d\lambda = \frac{8\pi hc \ \lambda^{-5} d\lambda}{e^{\frac{hc}{\lambda kT}} - 1}$$

This shows that energy density u at any wavelength λ is enough for the determination of temperature T. But the radiation actually will be a spectrum, so to measure temperature we require measurement over a range of wavelengths.

The first experimental evidence for the existence of this microwave radiation was obtained in the year 1965. In this year two American physicists at the Bell telephone laboratories in New Jersey Arno Penzias and Rober Wilson, used a microwave antenna tuned to a wavelength of 7.35 cm. At this wavelength they recorded an annoying hiss from their antenna that could not be eliminated, no matter how much care they took in refining the measurements. After pains taking efforts to eleminate to the noise they concluded that it was coming from no identifiable source and was striking their antenna from all directions, day and night, summer and winter. From the radiant energy at that wavelength they deduced a temperature of $3.1 \pm 1.0K$. It was later concluded that the radiation was nothing but the microwave background radiation, the remnants of Bing-Bang. For this discovery Penzias and Wilson shared the Nobel Prize in Physics.

The temperature of the back ground radiation deduced by Penzias and Wilson was not that much accurate. This is because no precise data were available below a wavelength of 1 cm due to atmospheric absorption. After that several sophisticated and refined experiments have conducted. The most recent measurements were made with the Cosmic Background Explorer (COBE) satellite, which was launched into earth orbit in the year 1989 and the Wilkinson Microwave Anisotropy (WMAP) satellite which was launched into solar orbit in the year 2001. The COBE and WMAP satellites were able to obtain very precise data on the intensity of the background radiation in the wavelength range between 1 cm and 0.05 cm. The results from the COBE satellite was plotted shown in figure 2.6. The data points fall precisely on the solid line which is calculated from equation (2) for a temperature of T = 2.725 K. For this remarkable result achieved by John Mather and George smoot were awarded the 2006 Nobel Prize in Physics.

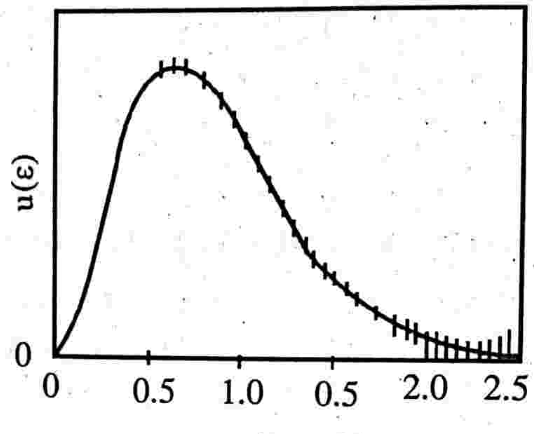


Figure 2.6

$$u(\upsilon) = \frac{8\pi h\,\upsilon^3}{c^3\left(e^{\frac{h\upsilon}{kT}} - 1\right)}$$

Put $h\upsilon = \varepsilon$

$$u(\varepsilon) \frac{8\pi\varepsilon^3}{h^2c^3\left(e^{\frac{\varepsilon}{kT}} - 1\right)}$$

Using the deduced value of T = 2.7K in equation (4), we can see that there are about 4.0×10^8 photon /m³. The average energy per photon is

$$\bar{E} = 2.325 \times 10^{-4} \times 2.7\text{eV}$$

$$\bar{E} = 6.3 \times 10^{-4}\text{eV}$$

Calculating the number of photons of the back ground radiation plays a vital role in Bing Bang cosmology. This is because after the Bing Bang the ratio of nucleaons (protons and neutrons) to photons is found to be almost constant till late. (after nearly 14×10^9 years).

Example 4

Evaluate the integral $\int_0^\infty \frac{x^2 dx}{e^x - 1}$ given $\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{\pi^3}{25.76}$

Solution

$$I = \int_0^\infty \frac{x^2 dx}{e^x - 1} = \int_0^\infty \frac{x^2 dx}{e^x(1 - e^{-x})}$$

$$I = \int_0^\infty \frac{e^{-x} x^2 dx}{1 - e^{-x}} = \int_0^\infty e^{-x} x^2 (1 - e^{-x})^{-1} dx$$

Using

$$(1 - x)^{-1} = \sum_{n=0}^{\infty} x^n, \quad x < 1$$

$$I=\int_0^{\infty} e^{-x}x^2\sum_{n=0}^{\infty} e^{-nx}$$

$$I=\sum_{n=0}^{\infty}\int_0^{\infty} e^{-(n+1)x}x^2dx$$

Using the standard integral $\int_0^{\infty} e^{-ax}x^n dx = n!(a)^{-n-1}$

$$\therefore \quad I=\sum_{n=0}^{\infty} 2!(n+1)^{-2-1}$$

$$I=2\sum_{n=0}^{\infty}\frac{1}{(n+1)^3}=2\sum_{n=1}^{\infty}\frac{1}{n^3}$$

$$I=2.\frac{\pi^3}{25.76}=2.404$$

Dark Matter

A galaxy is a huge collection of nebulae (gases), stellar remnants and billions of stars and their solar system held together by gravity. There are six types of galaxies. They are spiral, elliptical, active, dwarf and irregular galaxies. Many galaxies have spiral structure with a bright central regions containing most of the galaxies mass and several arms in a flat disc. The entire structure rotate about an axis perpendicular to the plane of the disc. We belong to milky way (spiral) galaxy, where sun is in one of the spiral arms at a distance of 8.5 kpc from the centre and has a tangential velocity of 220 kms^{-1}. At this speed it takes about 240 million years for a complete rotation. During the life time of the solar system of about 4.5 billion years, the sun has made about 20 revolutions.

To analyse the motion of stars, in the galaxy we can make use of Kepler's harmonic law of motion $(T^2 \alpha r^3)$ because stars in the galaxy are bound due to gravitational force. We assume that the gravitational force on the sun is due to masses at the centre of the galaxy. The other stars in the spiral arm, whose mass contribution to the force on the sun is negligibly small. According to Kepler's third law, we have

$$T^2=\frac{4\pi^2}{GM}r^3$$

where M is the mass contained within region of radius r.

Using $T = \frac{2\pi r}{v}$ we get

$$v = \sqrt{\frac{GM}{r}}$$

Put $v = 220 kms^{-1} = 220 \times 10^3 ms^{-1}$

$r = 8.5 kpc = 8.5 \times 10^3 \times 3.08 \times 10^{16} = 26.18 \times 10^{19} m$

$$\therefore \quad M = \frac{v^2 r}{G} = \frac{(220 \times 10^3)^2 \times 26.18 \times 10^{19} m}{6.67 \times 10^{-11}}$$

$$M = \frac{22^2 \times 10^8 \times 26.18 \times 10^{19}}{6.67 \times 10^{-11}}$$

$$M = \frac{22^2 \times 26.18}{6.67} \times 10^{38} kg$$

$$M = 1.9 \times 10^{41} kg$$

$$M_\odot \simeq 1.9 \times 10^{30} kg$$

$$M = \frac{1.9 \times 10^{41}}{1.9 \times 10^{30}} \approx 10^{11} \text{ solar mass.}$$

This shows that a mass equivalent of 10^{11} solar masses lies within the Sun's orbit.

According to this model, we expect stars beyond the Sun to have tangential velocity that decreases with r. Within the solar system all planets exactly and precisely follow this model. However we observe that v is almost constant or perhaps increases slightly for stars beyond the Sun. See figure below. Other galaxies also show the same effect. This shows that either the model is not correct or there is something missing.

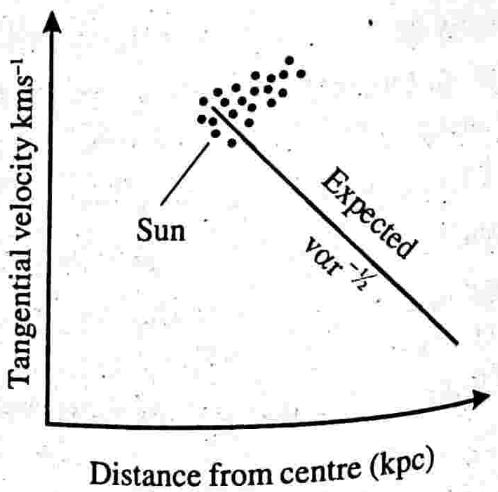


Figure 2.7

The tangential speeds of stars in distant galaxies can be measured by the Doppler shift of their light. If we are viewing a galaxy along the plane of the disc, then one side will always be moving towards us and the other will always be moving away from us. From the difference between the Doppler shifts of the light from the two sides of the galaxy we can calculate the rotational speed from this measurement, we can determine how the tangential velocity of the galaxy depends of the distance from its centre it is observed that the tangential velocity found to be constant throughout the visible part of the galaxy. See figure 2.8.

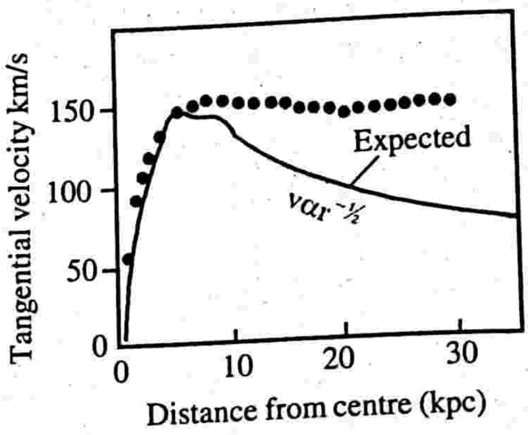


Figure 2.8

In other words the results obtained are not consistent with Kepler's law. In order to maintain the status quo of Kepler's law as well as the observational evidence that v is independent of r, there is a simple way. if we assume that M is a linear function of r, both are satisfied. But this assumption is not correct because most of the masses are concentrated in the central region of galaxy.

To resolve this it has been concluded that there is a large quantity of invisible matter in galaxy which contributes to gravitational force. This invisible matter is called dark matter. To supply the required gravitational force this dark matter must have atleast 10 times the mass of the visible matter in the galaxy. That is more than 90% of the matter in the galaxy is in the invisible form.

Dark Matter

Dark matter is a term used to describe the invisible form of matter that can be inferred to exist from its gravitational effects, but does not emit or absorb detectable amount of light.

The adjective dark is used since it (almost) neither emits nor absorbs electromagnetic radiations.

According to the present cosmological evidence, **dark matter is that something which provides the gravitational attractive force that keeps this together and explains how the cohesion of stars, galaxies and even the galactic clusters is possible.**

The evidence for the existence of dark matter comes from the observation of light from distant galaxies that passes by a cluster of galaxies on its way to earth. This light is deflected by the gravitational field of the cluster in a process known as gravitational lensing. From the amount of deflection of the light, it is possible to deduce the quantity of matter in the cluster. The results of these observations show that there is much more matter in the cluster than we would deduce from the luminous matter alone provide evidence for the existence of dark matter.

Finally we discuss what kind of objects make up this dark matter. There two speculation in this regard. One is that dark matter is a Massive Compact Halo Objects (MACHO) which includes black holes, neutron stars, white dwarfs, brown dwarfs etc. The other is that dark matter is a Weakly Interacting Massive Particles(WIMPS) which include neutrinos, magnetic monopoles etc. The major difference between the two types of objects is that MACHOs are made from baryons while WIMPS are non-baryonic matter current theories suggest that the most of the dark matter is of the non-baryonic form. But no examples of this types of matter have yet been produced in any laboratory on earth except neutrions.

It is not nice to go without mentioning dark energy though it is not in the syllubs.

Dark energy

It is a hypothetical form of energy that permeats all space and enters a negative pressure, so as the universe expands, the pressure increase and causes the universe to expand at an ever increasing rate.

Dark energy is a kind of repulsive gravity, actually pushing the universe apart. The effect of dark energy is small for objects of the size of galaxies and stars but is critical for understanding the large scale structure of the universe.

Are dark matter and dark energy related

It is natural to conjecture the dark matter and dark energy as two different manifestations of the same physical quantity according to Einstein's mass energy equivalence. But the two do not seem to be related to each other.

Dark matter is the force that keeps the universe together and explains how the cohesion of stars, galaxies and even the galactic clusters is possible. Dark energy, on the other hand, is the force responsible for the acceleration of the expansion of the universe at an ever increasing rate. The influence of dark matter is attractive where as that of dark energy is repulsive. Thus dark matter and dark energy are two competing forces in our universe.

Cosmology and general relativity

So far we have been dealing with general theory of relativity and its predictions. Now we discuss another area where general relativity has proved to be useful almost from its inception. This is **the branch of astronomy dealing with the large scale structure of the universe as a whole: briefly called cosmology.**

Cosmology deals with the study of the universe on the large scale, including its origin, evolution and future. For this study we require relativity (both), quantum theory, the fundamental results from atomic and molecular physics, statistical physics, thermodynamics, nuclear physics and particle physics.

In general relativity the governing equation is the Einstein's field equation, which reads

$$\text{Curvature of space} = \frac{8\pi G}{c^4}\,\text{mass energy density.}$$

This equation describes the entire universe. Since our aim is to draw cosmological predictions, we are not interested in the local variations in energy density but rather in the large scale variations of the energy density. The average density of the entire universe over a distance that is large compared with the spacing between galaxies is called large scale variations in the density.

It is due to the expansion of the universe, the density of the universe is not a constant i.e, density ρ is a function of time. Solving Einstein's field equation for the large scale structure of the universe we get

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi}{3}G\rho R^2 - Kc^2$$

This is known as Friedmann equation. Here R(t) represents the size of the universe at time t, ρ represents the mass-energy density of the universe at time t. K is a constant which gives geometrical structure of the universe.

For $K = 0$, the universe is flat

$K = 1$, the universe is curved and closed

$k = -1$, the universe is curved and open.

To solve Friedmann equation we have to specify the value of K. On the large scale; our universe seems to be flat, so we take $K = 0$.

$$\therefore \quad \left(\frac{dR}{dt}\right)^2 = \frac{8\pi}{3}G\rho R^2 \qquad \text{..... (1)}$$

The mass energy density ρ includes both the matter and radiation present in the universe. But the present universe is dominated by matter and the contribution of radiation to ρ is negligible.

$$\rho_m = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3}$$

As M is constant $\rho_m \ \alpha \ R^{-3}$

Putting this value in the above equation (1), we get

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi}{3} G \ \frac{M}{\frac{4}{3}\pi R^3} R^2$$

$$\left(\frac{dR}{dt}\right)^2 = 2\frac{GM}{R}$$

or $$\frac{dR}{dt} = \frac{\sqrt{2GM}}{R^{\frac{1}{2}}}$$

$$R^{\frac{1}{2}} dR = \sqrt{2GM}\ dt$$

Integrating, we get

$$\frac{R^{\frac{3}{2}}}{\frac{3}{2}} = \sqrt{2GM}\ t$$

or $$R^{\frac{3}{2}} = \frac{3}{2}\sqrt{2GM}\ t$$

$$R(t) = \left(\left(\frac{3}{2}\right)\sqrt{2GM}\right)^{\frac{2}{3}} t^{\frac{2}{3}} = At^{\frac{2}{3}}$$

$\therefore$ $$\frac{dR}{dt} = \frac{2}{3} At^{\frac{-1}{3}}$$

Putting the value of R(t) and $\frac{dR}{dt}$ in equation (1), we get

$$\left(\frac{2}{3}A\,t^{\frac{-1}{3}}\right)^2 = \frac{8\pi}{3}G\rho_m\,A^2\,t^{4/3}$$

$$\frac{4}{9}A^2\,t^{\frac{-2}{3}} = \frac{8\pi}{3}G\rho_m\,A^2\,t^{4/3}$$

$\therefore$ $\quad t = \dfrac{1}{\sqrt{6\pi G\rho_m}}$. This is the expression for the matter dominated universe.

In contrast, the early universe was dominated by radiation, the contribution to ρ from matter is negligible. Here radiation density $\rho_r \;\alpha\; R^{-4}$. This is because from Plank's radiation law

energy density $u \;\alpha\; \dfrac{d\lambda}{\lambda^5}$

when all the wavelengths scale with R $d\lambda \;\alpha\; R$ and $\lambda^5 \;\alpha\; R^5$

so $\rho_r \;\alpha\; R^{-4}$

proceeding as before we get

$$R(t) = A'\,t^{\frac{1}{2}}$$

and $\quad t = \sqrt{\dfrac{32}{\pi G\rho_r}}$. This is the expression for the radiation dominated universe.

Now we introduce the Hubble parameter which can be defined as time variation of the scale factor R.

i.e., $\quad H = \dfrac{1}{R}\dfrac{dR}{dt}$

If the universe has been expanding at a constant rate, i.e., $R \;\alpha\; t$

or $\quad R = Bt$

$$\frac{dR}{dt} = B$$

$\therefore$ $\quad H = \dfrac{1}{R}\dfrac{dR}{dt} = \dfrac{B}{Bt} = \dfrac{1}{t}$

or $\quad t = H^{-1}$

In this case the age of the universe is the reciprocal of the Hubbles parameter. For the matter dominated universe, we have

$$R(t) = A\, t^{2/3}$$

$$\frac{dR}{dt} = A \cdot \frac{2}{3}\, t^{-1/3}$$

$$\therefore \quad H = \frac{1}{R}\frac{dR}{dt} = \frac{A \cdot \frac{2}{3}\, t^{-1/3}}{A\, t^{2/3}}$$

$$H = \frac{2}{3t}$$

or $$t = \frac{2}{3} H^{-1}$$

For the radiation dominated universe, we have

$$R(t) = A^{1/2}\, t^{1/2}$$

we get $$t = \frac{1}{2} H^{-1}$$

This shows that in either case we can take H^{-1} as rough estimate measure of the age of the universe at any time.

The above discussion shows that we can characterise the universe by several parameter such as K, ρ, R(t) and H. The challenge to the observational astronomer is to obtain data on the distribution and motion of the stars and galaxies that can be analysed to obtain values for the above parameters.

The big bang cosmology

Our present universe is characterised by a relatively low temperature (2.7K) and a low density of particles. In other words present universe is matter dominated. This structure and evolution of the universe are dominated by gravitational force. As our universe is expanding and cooling, in the distant past universe must have been characterised by a higher temperature and greater density of particles. Let us imagine that we go back in time and examine the universe at earlier times, even before the formation of stars and galaxies. At some time in the history, the temperature of universe must have been high to ionise atoms. At that time universe consisted of a plasma of electrons and positive ions and electromagnetic force was important in determining the structure of the universe. As we go back again in time the temperature of the universe must have been enough to ionise nuclie. This time universe consisted of electrons, protons and neutrons along with radiation. In this time the strong nuclear force was important in determining the evolution of the universe. At still earlier times the weak interaction played a significant role.

If we try to go back in time still further universe consists of leptons and quarks. In this time quarks, leptons and radiations were important in determining the evolution of the universe. Since we do not know much about quarks and their interaction. So we can't describe this very early state of the universe. Quarks and their interaction are now not subjects of particle physicists and they are in the hot pursuit of discovering it. The signatures of quarks have been observed at Geneva-Switzerland some year back. We now strongly believe that one day we will be able to understand the interactions and their properties completely, then we can go back still in earlier times. **Eventually we reach a fundamental barrier when the universe had an age of 10^{-43}s, which is known as Planck time. None of the theories in physics gives us any clue about the structure of the universe before Planck time.**

Later than Planck time but before the condensation of bulk matter the universe consisted of particles, antiparticles and radiation was almost in thermal equilibrium temperature T. The universe at this time is radiation dominated. To evaluate the temperature T at this time we make use of the relation.

$$t = \sqrt{\frac{3}{32\pi G\rho_r}}$$

Using $\rho_r = \frac{u}{c^2}$ and $u = \frac{8\pi^5 k^4}{15\, c^3 h^3} T^4$

or $$\rho_r = \frac{8\pi^5 k^4 T^4}{15\, c^5 h^3}$$

$$\therefore \quad t = \sqrt{\frac{3 \times 15 c^5 h^3}{32\pi G\, 8\pi^5 k^4 T^4}}$$

$$t = \sqrt{\frac{45}{256}\,\frac{c^5 h^3}{G\pi^6 k^4}}\cdot\frac{1}{T^2}$$

$$\therefore \qquad \left(\frac{45}{256}\frac{c^5h^3}{G\pi^6k^4}\right)^{1/4}\cdot\frac{1}{t^{1/2}}=\frac{A}{t^{1/2}} \text{ kelvin}$$

substituting the values of c, h, G and k

we get $\quad A \simeq 1.5\times10^{10}$

$$\therefore \qquad T=\frac{1.5\times10^{10}}{t^{1/2}}K$$

The equation relates the age of the early universe to its temperature.

The radiation of the early universe consisted of high energy photons, whose average energy is estimated as kT. The interactions between the radiation and matter can be represented by our familiar processes.

Photons $\rightarrow$ particle + antiparticle

Particle + Antiparticle $\rightarrow$ Photons

For example $\quad 2\gamma \rightarrow e^+ + e^-$

$$e^+ + e^- \rightarrow 2\gamma$$

From conservation of energy and momentum

$$2m_ec^2 = 2h\upsilon$$

or $\quad m_ec^2 = h\upsilon = 0.511$ MeV, minimum value of photon

Example 5

At what age did the universe cool below the threshold temperature for (a) nucleon production (b) pi-meson production. $m_p = 1.67\times10^{-27}$ kg and $m_\pi = 140$MeV

Solution

a) To calculate the threshold frequency, we can make use of

$$mc^2 = kT$$

$$T=\frac{mc^2}{k}=\frac{1.67\times10^{-27}\times(3\times10^8)^2}{1.38\times10^{-23}}$$

$$T=\frac{1.67\times9\times10^{-11}}{1.38\times10^{-23}}=1.09\times10^{13}K$$

Using $T = \dfrac{1.5\times10^{10}}{t^{1/2}}$

$$t = \frac{(1.5\times10^{10})^2}{T^2} = \frac{2.25\times10^{20}}{1.09^2\times10^{26}}$$

$$t = 1.89\times10^{-6}\,s$$

b) The mass of pi-meson $m_\pi = 140$ MeV

$$\therefore \qquad T = \frac{mc^2}{k} = \frac{140\times10^6\,eV}{8.617\times10^{-5}\,eV/K}$$

$$T = 1.62\times10^{12}\,K$$

Age of the universe $t = \dfrac{(1.5\times10^{10})^2}{(1.62\times10^{12})^2}$

$$t = \frac{2.25\times10^{20}}{1.62^2\times10^{24}} = \frac{2.25}{1.62^2}\times10^{-4}\,s$$

$$t = 8.57\times10^{-5}\,s$$

Example 6

At what temperature was the universe hot enough to permit the photons to produce K mesons (500 MeV) ? At what age did the universe have this temperature.

Solution

$$T = \frac{mc^2}{k} = \frac{500\times10^6\,eV}{8.617\times10^{-5}\,eVK^{-1}} = 5.8\times10^{12}\,K.$$

$$t = \left(\frac{1.5\times10^{10}}{5.8\times10^{12}}\right)^2 = 6.68\times10^{-6}\,s$$

We begin to analyse the evolution of the universe from $1\mu s$ (age of the universe)

1) At $t = 1\mu s$

The temperature of the universe can be calculated to be $T = \dfrac{1.5\times10^{10}}{t^{1/2}}\,K$

i.e., $$T = \frac{1.5\times10^{10}}{(10^{-6})^{1/2}} = 1.5\times10^{13}\,K$$

The energy of the particles is = kT

$$= 8.617\times10^{-5}\times1.5\times10^{13}\,eV$$

$$\simeq 1300\,MeV$$

At present the temperature of the background radiation is 2.7K and the observable radius of the universe is 10^{26} m. At 1μs, the temperature is raised to 1.5×10^{13}K.

At 2.7 K, the size is 10^{26} m

At 1K, the size is $10^{26}\times2.7$

At 1.5×10^{13}K, the size is $\frac{10^{26}\times2.7}{1.5\times10^{13}}$

$$=1.8\times10^{13}\,m$$

So the size of the universe at 1μs is about the present size of the solar system (10^{13}m). At energy about 1300MeV, the particle present are p, $\bar{p}$, n, $\bar{n}$, e^-, e^+, μ^+, $\bar{\mu}$, π^0, π^-, π^+, photons, neutrinos and antineutrinos etc. So particle creation and annihilation may occur, so the number of particles is almost equal to the number of antiparticles. Further, the number of photons is roughly equal to the number of nucleons, which in turn roughly equal to the number of electrons. But the relative number of protons and neutrons is determined by three factors.

(i) Boltzmann factor $e^{-\frac{\Delta E}{kT}}$ decides which one is larger in number.

$$\Delta E = (m_n - m_p)c^2 = (939.56 - 938.27)\,MeV \simeq 1.3\,MeV$$

$$kT = 8.617\times10^{-15}\frac{eV}{K}\cdot1.5\times10^{13}K = 13\times10^{8}\,eV$$

$$kT = 1300\,MeV.\ \text{So}\ e^{-\frac{\Delta E}{kT}} = e^{-\frac{1}{1000}} = 0.999 = 1$$

Thus number of the protons is almost equal to the number of neutrons.

(ii) The nuclear reactions such as $n+\upsilon_e \rightleftharpoons p+e^-$, $n+e^+ \rightleftharpoons p+\bar{\upsilon}_e$ neutrons and protons are produced in both directions as long as there are plenty of e^-, e^+, υ_e and $\bar{\upsilon}_e$

(iii) The neutron decay time is about 10 minutes. So when $t < 1s$, there is not enough time for the neutron to decay. All the above three factors keep the neutron to proton ratio close to 1.

2) At $t=10^{-2}s$ we can calculate $T=1.5\times10^{11}K$ and $kT=13MeV$. This shows that photons have too little energy to produce pions and muons and because of pion and muon life times are much shorter than $10^{-2}s$, they might have decayed into electrons, positrons and neutrinos. At this energy pair production of nucleons and antinucleons no longer occurs, but nucleon-antinucleon annihilation continues. But pair production of electrons and positrons can still occur. So at $t=10^{-2}s$, universe consists of p, n, e^-, e^+, photons and neutrinos. The neutron to proton ratio remains almost about 1.

3) At $t=1s$

The temperature of the universe becomes $T=1.5\times10^{10}K$ and $kT=1.3MeV$.

The neutron - proton ratio

$$\frac{N_n}{N_p}=e^{-\frac{\Delta E}{kT}}=e^{-\frac{1.3MeV}{1.3MeV}}=e^{-1}=0.37$$

$\therefore$ The relatative number of protons $\frac{N_p}{N_p+N_n}=\frac{1}{1+\frac{N_n}{N_p}}=\frac{1}{1+0.37}=0.73$

This shows that there are 73% protons and 27% neutrons. During this period the influence of the neutrinos has been decreasing. To convert a proton to a neutron by capturing an antineutrino $(\bar{\upsilon}_e+p\rightarrow n+e^+)$, antineutrino having energy atleast 1.8 MeV above the mean neutrino energy (1.3MeV) at this temperature. This begins the time of neutrino decoupling. Since the capturing of neutrinos no longer occur, it begins to fill the entire universe and neutrinos cool along with the expansion of the universe. These primordial neutrinos presently have almost the same density of microwave photons but a slightly lower temperature t = 2K.

4) At $t = 6s$

Now the temperature of the universe becomes $T = \frac{1.5\times10^{10}}{6^{1/2}} K = 6.12\times10^{9} K$ and the average energy

$$kT = 8.617\times10^{-5}\times6.12\times10^{9} = 4.99\times10^{-5}$$
$$= 0.499 \text{ MeV}$$

This energy is not sufficient to produce electron-positron pair. But electron-positron annihilation continues and almost all positrons and 99.999% electrons ares are annihilated. Since the electrons have too little energy (0.499 MeV), conversion of proton to neutron cannot occur. However the weak interaction process that influences the relative number of protons and neutrons is the radioactive decay of the neutron. Since the decay time of neutron is 10 minutes this will not occur. Now there are about 84% protons and 16% neutrons. i.e., proton number is about 5 times the neutrons at this stage there are no remaining positrons or antinucleons because particle-anti particle annihilation has reduced the number of nucleons while the photon number remains the same and there are about $10^{9}N$ photons and the same number of neutrinos.

i.e., at $t = 6s$

The number of protons $= N$

The number of electrons $= N$

The number of neutrons $= \frac{N}{5}$

The number of photons $= 10^{9}N$

The number of neutrions $= 10^{9}N$

The formation of nuclei and atoms

In this section we discuss at what temperature and age of the universe, the formation of nuclei and atoms begin to occur.

We found that the age of the universe goes from 10^{-6}s to 6s all the time protons, neutrons and photons were present. suppose neutrons and protons collide it is possible to form deuterons (^{2}H).

i.e., $n + p \rightarrow {}^{2}H + \gamma$

and also $\gamma + {}^{2}H \rightarrow n + p$.

we know that the binding energy of deuteron is 2.22 MeV. Thus in order to form deuterons the energy of photons must be less than 2.22 MeV. If the photons have energy greater than 2.22 MeV it will break up the deuterons formed. The temperature corresponding to 2.22 MeV is,

$$T = \frac{2.22\text{MeV}}{8.617\times10^{-5}}\text{eVK}^{-1}$$

$$T = \frac{2.22\times10^{6}}{8.617\times10^{-5}} = 2.57\times10^{10}\text{K}$$

$$T \simeq 2.5\times10^{10}\text{K}$$

This shows that when the temperature falls below 2.5×10^{15}K stable deuterons will be formed. However this does not occur. This is because photons do not have single energy.

The photon energy distribution is a spectrum. A small fraction of photons has energies above 2.22 MeV and this will continue to break the deuterons. See figure 2.9.

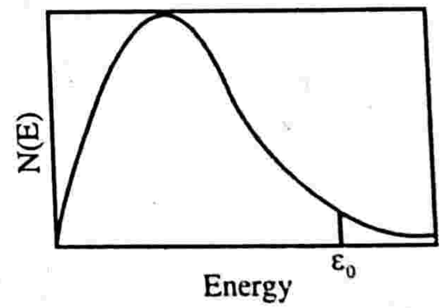


Figure 2.9: The thermal radiation spectrum. The photons above $\varepsilon_0 = 2.22$ MeV are sufficiently energetic to break apart deuterium unclei

Before matter and antimatter annihilation occured, there were about as many photons as nucleons and antinucleons, but after $t = 10^{-2}$s the ratio of nucleons to photons is about 10^{-9}. Out of these about $\frac{1}{6}\times10^{-6}$ of the nucleons are neutrons. If the fraction of photons above 2.22 MeV is greater than $\frac{1}{6}\times10^{-6}$, there will be atleast one energetic photon per neutron, which effectively prevents deuteron formation. Our next aim is to calculate to what temperature the photons must cool before fewer than $\frac{1}{6}\times10^{-9}$ of them are above 2.22 MeV.

We have the relation

$$n(\upsilon)d\upsilon = \frac{8\pi\upsilon^2 V d\upsilon}{c^3}\frac{1}{e^{h\upsilon/kT}}-1$$

$$\frac{n(\upsilon)d\upsilon}{V} = \frac{8\pi\upsilon^2 d\upsilon}{c^3 e^{h\upsilon/Kt} - 1}$$

Put $h\upsilon = \varepsilon$, $hd\upsilon = d\varepsilon$

$$\frac{n(\upsilon)d\upsilon}{V} = \frac{8\pi\,\varepsilon^2\,d\varepsilon}{h^3c^3} \quad \frac{1}{e^{\frac{\varepsilon}{kT}} - 1}$$

since $\quad \varepsilon \gg kT, \ \dfrac{1}{e^{\frac{\varepsilon}{kT}} - 1} \simeq e^{-\frac{\varepsilon}{kT}}$

$$\frac{n(\upsilon)d\upsilon}{V} = \frac{8\pi\,\varepsilon^2}{h^3c^3}\, e^{-\frac{\varepsilon}{kT}} d\varepsilon$$

Integrating this we get the total number density

$$\frac{N}{V} = \int_0^\infty \frac{n(\upsilon)d\upsilon}{V} = \int_0^\infty \frac{8\pi\,\varepsilon^2}{h^3c^3}\, e^{-\frac{\varepsilon}{kT}} d\varepsilon$$

$$\frac{N}{V} = \int_0^{\varepsilon_0} \frac{8\pi\,\varepsilon^2\, e^{-\frac{\varepsilon}{kT}} d\varepsilon}{h^3c^3} + \int_{\varepsilon_0}^\infty \frac{8\pi}{h^3c^3}\varepsilon^2 e^{-\frac{\varepsilon}{kT}} d\varepsilon$$

Where $\varepsilon_0 = 2.22$ MeV

Thus

$$\left(\frac{N}{V}\right)_{\varepsilon > \varepsilon_0} = \frac{8\pi}{h^3c^3}\int_{\varepsilon_0}^\infty \varepsilon^2 e^{-\frac{\varepsilon}{kT}} d\varepsilon$$

Integrating by parts we get

$$\left(\frac{N}{V}\right)_{\varepsilon > \varepsilon_0} = \frac{8\pi}{h^3c^3} k^3T^3\, e^{-\frac{\varepsilon_0}{kT}} \left[\left(\frac{\varepsilon_0}{kT}\right)^2 + \frac{2\varepsilon_0}{kT} + 2\right]$$

Recall our expression for $\dfrac{N}{V} = \dfrac{19.23\,\pi\,k^3\,T^3}{c^3\,h^3}$

$$\therefore \quad \frac{\left(\frac{N}{V}\right)_{\varepsilon>\varepsilon_0}}{\left(\frac{N}{V}\right)} = \frac{8}{19.23} e^{-\frac{\varepsilon_0}{kT}}\left[\left(\frac{\varepsilon_0}{kT}\right)^2 + 2\frac{\varepsilon_0}{kT} + 2\right]$$

$$= 0.42\, e^{-\frac{\varepsilon_0}{kT}}\left[\left(\frac{\varepsilon_0}{kT}\right)^2 + 2\frac{\varepsilon_0}{kT} + 2\right]$$

To prevent deuteron formation, the fraction must be $\frac{1}{6}\times 10^{-9}$

$$\therefore \quad \frac{1}{6}\times 10^{-9} = 0.42 e^{-\frac{\varepsilon_0}{kT}}\left[\left(\frac{\varepsilon_0}{kT}\right)^2 + 2\frac{\varepsilon_0}{kT} + 2\right]$$

Put $\quad \frac{\varepsilon_0}{kT} = x$

$$\frac{1}{6}\times 10^{-9} = 0.42 e^{-x}(x^2 + 2x + 2)$$

or $\quad e^x = 6\times 0.42\times 10^9 (x^2 + 2x + 2)$

$$e^x = 2.52\times 10^9 (x^2 + 2x + 2)$$

Solving this we get $x = 28$

Thus, $\quad \frac{\varepsilon_0}{kT} = 28$

$$T = \frac{\varepsilon_0}{k\times 28} = \frac{2.22\text{MeV}}{8.617\times 10^{-5}\frac{\text{eV}}{\text{K}}\times 28}$$

$$T = \frac{2.22\times 10^6}{8.617\times 10^{-5}\times 28} = \frac{222\times 10^9}{8.617\times 28}\text{K}$$

$$T = 9.2\times 10^8\text{K} = 9\times 10^8\text{K}$$

This shows that when $T > 9\times10^8 K$, the number of photons with $\varepsilon > 2.22 MeV$ the deuteron formation is prevented. When $T < 9\times10^8 K$ deuterons are produced.

The age of the universe corresponding to this temperature

$$t = \left(\frac{1.5\times10^{10}}{T}\right)^2$$

$$t = \left(\frac{1.5\times10^{10}}{9\times10^8}\right)^2 = 277s \simeq 250s$$

This shows that after about 250s deuterons begin to form. Then deuterons react with protons and neutrons available to give helium and tritium.

$$^2H + p \rightarrow {}^3He + \gamma$$

and $$^2H + n \rightarrow {}^3H + \gamma$$

The energies of the formation of these nuclei are 5.49 MeV and 6.26 MeV respectively. The reaction continues to give ^4He

$$^3He + n \rightarrow {}^4He + \gamma$$

and $$^3H + p \rightarrow {}^4He + \gamma$$

There are no stable nuclei with A = 5, so no further reactions of this sort are possible. It is also not possible to combine two helium ^4He to form beryllium ^8Be. Since Be is highly unstable. It would be possible to form stable ^6Li and ^7Li, but these are very small in quantities relative to H and He. From Li further reactions are possible such as $^7Li + {}^4He \rightarrow {}^{11}B$ and so forth, but these occur in still small quantities. The end products ^2H and He along with the left over original protons make up about 99.9999% of the nuclei after the era of nuclear reactions.

When the age of the universe is t = 250s, the original 16% neutrons present at t = 6s had beta-decayed to about 12%, leaving 8% protons. Most of the ^2H, ^3H and ^3He were transformed into heavier nuclei. So we can assume the universe to be composed mostly of ^1H and ^4He nuclei. Of the N nucleons present at t = 250s, 12% (0.12N) were neutrons and 0.88N were protons. The 0.12N neutrons combined with 0.12N protons forming 0.06N ^4He and leaving $0.88N - 0.12N = 0.76N$ protons.

The universe then consisted of 0.82N nuclei, of which 0.06N (7.3%) were ^4He and 0.76N (92.7%) were protons. Helium is about four times as massive as hydrogen. So by mass the universe is about 24% helium.

At this point the universe began long and uneventful period of cooling. during which the strong interactions ceased to be of importance.

The final step in the evolution of the primitive universe is the formation of neutral hydrogen and helium atoms from the 1 H, 2 H, 3He and 4He and the free electrons. In the case of hydrogen, this takes place when the photon energy drops below 13.6eV. Otherwise atoms will be ionised by the radiation. There are still about 10^9 photons for every proton and so we must wait for the radiation to cool until the fraction of photons above 13.6eV is less than about 10^{-9}. As before we can solve for T, for the fraction $F = 10^{-9}$. We get $\frac{\varepsilon_0}{kT} = 6$ with $\varepsilon_0 = 13.6\text{eV}$. $\therefore T = 6070K$ which occur at time $6.1 \times 10^{12} s = 190,0000$ years. This estimate is only a rough one, because for those calculation we considered only energy density of the radiation present in the universe. But as the universe cools, the contribution of matter to the energy density becomes more significant, so the temperature drops more slowly than we would estimate. This contribution may increase this time by about a factor o 2 to about 3,80,000 years, and the radiation temperature is decreased by about a factor of $\sqrt{2}$, to T = 4300 K.

After the formation of neutral atoms, there are virtually no charged particles left in the universe and the radiation left is not energetic enough to ionise the atoms. This is the time of decoupling of the radiation field from the matter and now the electro magnetism, the third of our basic forces is no longer important in shaping the evolution of the universe. This point onwards gravity plays its role to shape the universe.

The time after 380,000 years had been comparatively uneventful, from the point of view of cosmology. Density fluctuations of the hydrogen and helium triggered the condensation of galaxies and then first generation stars were born. Supernova explosions of the material from these stars permitted the formation of second generation systems, among which planets formed from the rocky debris.

Meanwhile, the decoupled radiation field, unaffected by the gravitational coming and going matter, began the long journey that eventually took it cooled again by a factor of 1600. Then everything could be observed by the radio telescope of 20th century.

The remarkable story of the evolution of the universe are summarised in figure below.

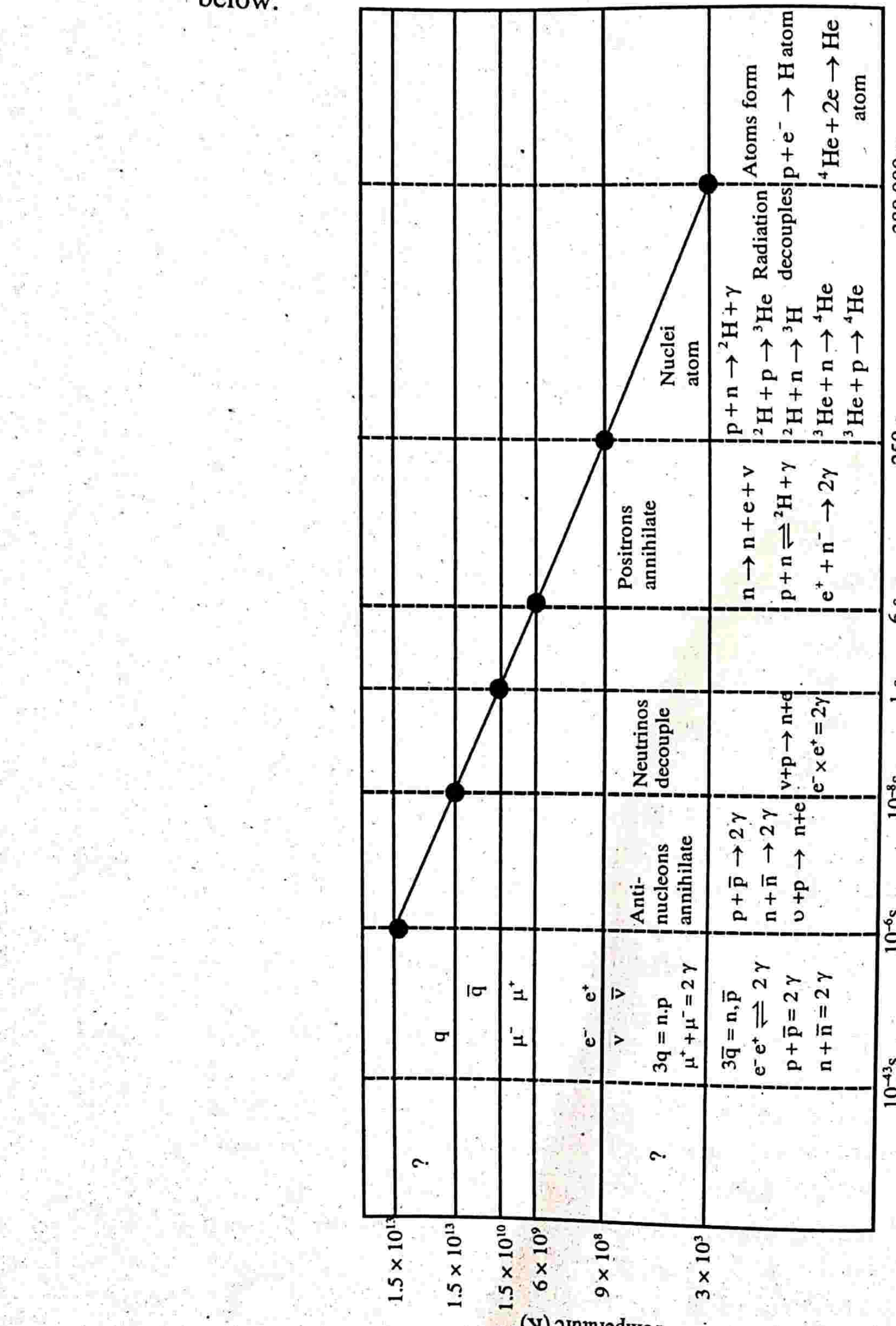


Figure 2.10: Evolution of the universe according to the Big Bang cosmology

IMPORTANT FORMULAE

1. Doppler shift in frequency of light

$$\frac{\upsilon' - \upsilon}{\upsilon} = \frac{\Delta v}{c}$$

2. Expression for frequency shift when a light wave falling in the earths gravity.

(i) $\frac{\Delta\upsilon}{\upsilon} = \frac{mgH}{mc^2}$

(ii) $\frac{\Delta\upsilon}{\upsilon} = -\frac{GM}{Rc^2}$

(iii) $\frac{\Delta\lambda}{\lambda} = \frac{GM}{Rc^2}$

3. Relation between space – time curvature and energy momentum.

Curvature of space = $\frac{8\pi G}{c^2}$ energy momentum.

4. Relativistic Doppler effect of light

$$\lambda' = \lambda\sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}$$

5. Hubble's law:

$V = H_0 d, \quad V = H_o = 72 kms^{-1} (Mpc)^{-1}$

6. Planck's radiation law:

$$n(\upsilon)d\upsilon = \frac{8\pi V}{c^3}\upsilon^2 \frac{d\upsilon}{e^{\frac{h\upsilon}{KT}} - 1}$$

$$u(\upsilon)d\upsilon = \frac{8\pi h}{c^3}\frac{\upsilon^3 d\upsilon}{e^{\frac{h\upsilon}{KT}} - 1}$$

$$u(\lambda)d\lambda = \frac{8\pi hc}{\lambda^5}\frac{d\lambda}{e^{\frac{h\upsilon}{\lambda KT}} - 1}$$

7. Average energy of the photon:

$$\bar{E} = \frac{8}{15 \times 9.23}\pi^4 kT$$

10. Friedmann equation:

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi}{3}G\rho R^2 - Kc^2$$

$K = 0$, the universe is flate

$K = 1$, the universe is curved and closed

$K = -1$, the universe is curved and open

11. For a flat universe – Matter dominated.

$$R(t) = At^{2/3}$$

12. Relation between Hubble parameter and size of the universe.

$$H = \frac{1}{R}\frac{dR}{dt}$$

13. For the radiation dominated universe.

$$R(t) = A^{1/2}t^{1/2}.$$

14. Rought estimate of the age of the universe.

$$t = \frac{1}{H}$$

15. Relation between age and temperature of the universe.

$$T = \frac{1.5\times10^{10}}{t^{1/2}}$$

16. Relation between mass of the particle produced and temperature of the universe.

$$mc^2 = kT$$

17. Boltzsmann factor $= e^{-\frac{\Delta E}{KT}} = \frac{N_n}{N_p}$

where $\Delta E = (m_n - m_p)c^2$

UNIVERSITY MODEL QUESTIONS

Section A

(*Answer questions in about* **two** *or* **three** *sentences*)

Short answer type questions

1. What is meant by principle of covariance?
2. What is meant by principle of equivalence?
3. What is general theory of relativity?
4. What were the predictions of general theory of relativity?
5. Briefly explain how accelerated frames can take into account the effect of gravity.

6. Write down the equation for frequency shift when a light wave falling in earth's gravity and explain the symbols used.
7. What is gravitational red shift?
8. How does gravity affect the space time in general relativity.
9. How does the motion of a particle occurs in space-time according to general theory of relativity.
10. Write down the relation between curvature of space and energy-momentum in general relativity.
11. Briefly explain how does Einstein's field equation become Newtons law of gravitation.
12. Distinguish between Newton's law of gravitation and Einstein's law of gravitation.
13. How does light bend in gravitational field?
14. What is cosmological constant?
15. Solving field equations Einstin got an expanding universe. What was the mistake he committed?
16. What were the two assumptions made by Friedmann to arrive at the conclusion that the whole universe is expanding?
17. How did we arrive at the conclusion that the universe is expanding?
18. What is red shift?
19. What is blue shift?
20. What is Hubble's law?
21. What is the relation between recessional speed and distance.
22. Demonstrate the expanding universe with an analogy.
23. What is cosmic microwave background radiation?
24. How did Gamow arrive at the prediction of the cosmic microwave background radiation.
25. What was the contribution of Penzias and Wilson with regard to microwave background radiation.
26. Name two satellites sent into the earths orbit to study microwave background radiation.
27. Is microwave background radiation a reality? Justify?
28. What is Bing-Bang.
29. What is the evidence of Bing-Bang.
30. What is dark matter?
31. What is the function of dark matter.
32 Distinguish between dark matter and dark energy.
33. What is cosmology.
34. What is Planck time.
35. Write down Friedmann equation and explain the symbols used.
36. Distinguish between matter dominated and radiation dominated universe.

37. What is bing-bang cosmology?
38. Write down the relation between age and temperature of the universe.
39. What is the connection between the formation of nuclei and atoms with age and temperature of universe.
40. What is the age of the universe when protons, neutrons were present?

Section B

(Answer questions in a paragraph of about half a page to one page)

Paragraph / Problem type questions

1. Explain how did Einstein arrive at the principle of equivalence.
2. Distinguish between S.T.R and G.T.R.
3. Derive an expression for gravitational frequency shift.
4. Distinguish between Newton's law of gravitation and Einstein's law of gravitation.
5. Explain three major prediations of general theory of relativity.
6. Explain the phenomenon of bending of light rays in the presence of gravitational field.
7. Explain the gravitational red shift and blue shift.
8. Show that inverse of Hubble's parameter is almost equal to the age of the universe.
9. Write a brief note on Friedmann universe and Einstein universe.
10. State and explain Hubble's law.
11. Explain how did Hubble arrived at the conclusion that the whole universe is expanding.
12. Write a brief note on COBE and WMAP.
13. The sun's mass is 2.0×10^{30} kg and its radius is 7.0×10^{-8} m. Find the approximate gravitational red shift in light of wavelength 500 nm emitted by the Sun. [1.06 pm]
14. Find the approximate gravitational red shift in 500 nm light emitted by a white dwarf star whose mass is that of the Sun but whose radius is that earth, 6.4×10^{6} m.
15. A satellite is in orbit at an altitude of 150 km, we wish to communicate with it using a radio signal of frequency 10^9 Hz. What is the gravitational change in frequency between a ground station and the satellite? Assume g is a constant. [1.63×10^{-2} Hz]
16. Light from a certain galaxy is red shifted so that the wavelength of one of its characteristics spectral lines is doubled. Assume the validity of Hubble's law, calculate the distance to this galaxy. [8.15×10^{3} ly]
17. From the expression for energy density of the thermal radiation.
    a) find the energy at which the maximum of the radiation energy spectrum occurs.
    b) Evaluate the peak photon energy of the 2.7K microwave background radiation.

$$\begin{bmatrix} E = 2.821kT \\ E = 6.64\times10^{-4}\text{eV} \end{bmatrix}$$

18. Evaluate $\int_0^\infty \frac{x^3}{e^x - 1}dx$ $\left[\frac{\pi^4}{15}\right]$

19. Calculate the age of the universe when deutrons were produced. Take $T < 9\times10^8 K$ [250 s]

20. What was the age of the universe when the nucleons consisted of 60% protons and 40% neutrons. Given that $\Delta E = (m_n - m_p)c^2 = 1.3 MeV$ [0.17 s]

Section C

(*Answer questions in about* **one** *or* **two** *pages*)

Long answer type questions

1. Derive an expression for the energy of the microwave background radiation.
2. Establish that the inverse of Hubbles parameter is a rough estimate of measure of the age of the universe.

Hints to solutions

13. $\frac{\lambda' - \lambda}{\lambda} = \frac{GM}{Rc^2} = \frac{6.67\times10^{-11}\times2\times10^{30}}{8} = 0.2117\times10^{-5}$

$\lambda' - \lambda = 0.2117\times10^{-5}\times500\times10^{-9} m$

$= 1.058\times10^{-12} m \simeq 1.06m$

14. $\lambda' - \lambda = \frac{GM}{Rc^2}\lambda = \frac{6.67\times10^{-17}\times2\times10^{30}\times500\times10^{-9}}{6.4\times10^6\times(3\times10^8)^2}$

$= 0.1157 nm$

15. $\frac{\Delta\upsilon}{\upsilon} = \frac{gH}{c^2} = \frac{9.8\times150\times10^3}{(3\times10^8)^2} = 16.33\times10^{-12}$

$\Delta\upsilon = 16.33\times10^{-12}\times10^9 = 1.633\times10^{-2} Hz.$

16. $\lambda' - \lambda = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$, $\lambda' = 2\lambda$ (given)

$2 = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \quad \because \; V = \frac{3}{5}c$

Using $v = H_o d$

$$d = \frac{v}{H_o} = \frac{3}{5}\frac{c}{H_o} = 8.15\times10^3\,\text{ly}$$

Take $H_o = 72\,\text{kms}^{-1}(\text{Mpc})^{-1}$

1 parsec = $3.08\times10^{16}\,\text{m}$

1 ly = $9.46\times10^{15}\,\text{m}$.

17. a) $u(\upsilon)d\upsilon = \frac{8\pi h}{c^3}\frac{\upsilon^3}{e^{\frac{h\upsilon}{KT}}-1}d\upsilon,\ \varepsilon = h\upsilon$

$$u(\varepsilon) = \frac{8\pi\varepsilon^3}{(hc)^3}\frac{1}{e^{\frac{\varepsilon}{kT}}-1}$$

Find $\frac{du}{d\varepsilon} = 0$, we get $e^{-x} = 1-\frac{x}{3}$ where $x = \frac{\varepsilon}{kT}$ gives $x = 2.841$

$\varepsilon = 2.821kT$

b) Use $\varepsilon = 2.821kT$

18. See example 4

19. Use $t = \left(\frac{1.5\times10^{10}}{T}\right)^2$

20. $\frac{N_p}{N_n} = \frac{0.6}{0.4} = 1.5$

Use $\frac{N_n}{N_p} = e^{-\frac{\Delta E}{KT}}$

$$\frac{\Delta E}{kT} = \ln(1.5) = 0.41$$

$$T = \frac{\Delta E}{0.41k} = \frac{1.3\text{MeV}}{0.41\times8.62\times10^{-5}\text{eVK}^{-1}}$$

$T = 3.7\times10^{10}\,\text{K}$

Then use $t = \left(\frac{1.5\times10^{10}}{T}\right)^2 = 0.17\text{s}$

UNIT THREE

3 BASIC TOOLS OF ASTRONOMY

Introduction

This chapter deals with some basics of astrophysics. Astrophysics is only an extension of classical astronomy in the same sense as quantum mechanics and nuclear physics are extensions of classical physics. What is astronomy? **Astronomy is the science which deals with (size, position, motion and composition) celestial objects and their phenomena.** Astronomy is a term originated from two Greek words astron and nomos. Astron means star and nomos means law. Then what is astrophysics? **Astrophysics is a branch of astronomy that deals with the physical properties of celestial objects such as luminosity, size, mass, density, temperature and chemical composition and their origin and evolution.** Actually astrophysics is a child born from the marriage of physics and astronomy. Astronomy is an observational science and not an experimental science like other sciences because we cannot control the condition of experiments or event which are occurring in heavenly bodies. In astronomy the events occur automatically in stars, galaxies and interstellar medium and we observe them from earth. Hence the repetition of an experiment in other sciences is replaced by statistical study of large samples and changes in experimental conditions are taken into account by observations of a large variety of closely similar objects. Statistics thus plays an important role in the astronomical method. **The main source of information about heavenly bodies is the study of electromagnetic radiations emitted from them.**

Although astronomy based on observations rather than experiments, it is a science in the strictest sense of the word as opposed to the superstitious subject of astrology. In ancient times the same individuals practiced astronomy and astrology simultaneously and they were considered as one and the same. But after the discovery of telescopes astronomy grew as an observational science where as astrology did not follow the method of science and grew with belief and establishes customs. As a result astronomy and astrology drifted poles apart. Astrology claims to predict the future of men from the prevalent planetary positions among the stars. The entire edifice of astrology is build upon the movement of nine planets (Sun, Moon, Mercury, Venus, Mars, Jupiter, Saturn, Rahu and Ketu) across the sky spread over an angular separation of 18° on either side of ecliptic. If we consider this 18° width spread across the sky as a ribbon, it is called zodiac. The planets rahu and ketu were called as dark planets because they were responsible for eclipses. Now a days we know that rahu and ketu are not planets but they are geometrical intersection points of apparent path of moon and ecliptic. It is not wise to put an end without saying the famous words of Kepler who was the astrologer of Wallenstein palace. Kepler once said "Astrology is the foolish daughter of wise mother astronomy".

Stellar distance

Our primary aim in astronomy is to determine the distance of distant objects in the sky. This is necessary because for the determination of many other parameter like luminosity, brightness etc. The basic tool required is distance. But determining distance in astronomy has always been and continues to be difficult and associated with errors. There are so many methods available, but till now there is no consensus about which method is the best. The Stellar parallax is the probably the most accurate especially for determining the distances to stars.

Stellar parallax method

Parallax

Let us first see the meaning of parallax. Hold a pencil P at a distance from your eyes. Look at the pencil by closing the right eye and then left eye. The position of the pencil seems to change with respect to the background. This apparent shift (angle) is called parallax. The distance between points of observation (here the distance between left eye and right eye) L and R is called basis.

Stellar parallax

Stellar parallax is a measure of the star's distance. it is basically the angular measurement when the star is observed from two different locations on the earth's orbit. These two positions are generally six months apart and so the star will appear to shift its position with respect to the more distant background stars. The parallax P of the star observed is equal to half the angle through which its apparent position appears to shift. The larger the parallax (p), the smaller the distance (d) to the star.

Stellar parallax method

To measure the distance of a star, the star is photographed from the position A of the earth against the background of the far off stars. Repeat the observation after six months from the position B of the earth on the other end of the baseline. See figure below. From the observations the half angle subtended by the baseline AB can be measured. This gives the parallax

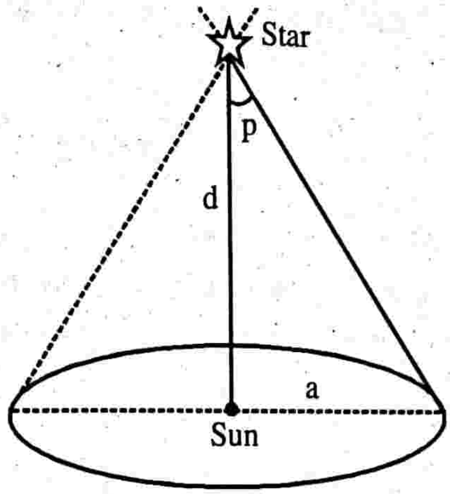


Figure 3.1

From the figure we have

$$\tan p = \frac{a}{d}$$

or $$p = \frac{a}{d} \qquad (\because \; p \text{ is very small})$$

$$d = \frac{a}{p} \qquad \text{..... (1)}$$

The distance $a\left(\frac{AB}{2}\right)$ is measured in metres. Usually the angle (p) is measured in radian. In the case of star the angle p is measured in seconds of arc. So we have to convert radian into seconds of arc.

We have $$1^\circ = \frac{\pi}{180} \text{ radian.}$$

$$1 \text{ radian} = \frac{180}{\pi} \text{ degree}$$

or $$1 \text{ radian} = \frac{180}{\pi} \times 60 \text{ minutes}$$

$$1 \text{ radian} = \frac{180}{\pi} \times 60 \times 60 \text{ seconds of arc}$$

$$1 \text{ radian} = \frac{180 \times 60 \times 60}{3.14158} \text{ seconds of arc}$$

$$1 \text{ radian} = 206,265'' \text{ seconds of arc}$$

$$\therefore \quad 1 \text{ second of arc} = \frac{1}{206265} \text{radian} = 4.848 \times 10^{-6} \text{ radian}$$

When $a = 1\text{AU} = 1.496 \times 10^{11}\text{m}$ and $p = 1''$

(1AU is the average distance from the earth to the Sun). Then the distance d is said to be one parallactic second (parsec). Parsec is symbolically written as pc

From eqn. 1, we have

$$1\,\text{pc} = \frac{1\text{AU}}{1''} = \frac{1.496 \times 10^{11}}{4.848 \times 10^{-6}}$$

$$1 \text{ pc} = 3.08 \times 10^{16}\text{m}$$

Definition of Parsec

One parsec is defined as the distance at which arc length one AU substends an angle of 1″.

From eqn. (1) we get

$$d = \frac{1}{p} \quad \text{When } a = 1$$

Now d is in parsecs and parallax p is in arc seconds. Thus, we can say the distance of star in parsecs in the reciprocal of its parallax p.

$$d = \frac{1}{p} \qquad \text{..... (2)}$$

For example if the measured parallax of a star is 0.01 arc second then the distance of the star is 100 pc.

Relation between AU, *l*y and pc

We have $1\text{AU} = 1.496 \times 10^{11}\text{m}$

This unit is used to measure very large distances within the solar system.

Light year is defined as the distance travelled by light in vacuum in one year.

$$s = vt = ct$$

$$1 \text{ light year} = 3 \times 10^8 \times 365 \times 24 \times 60 \times 60$$

i.e, $1\,ly = 9.46 \times 10^{15}\,m$

This unit is used for measuring large distances of stars and galaxies.

$$1pc = 3.08 \times 10^{16}\,m$$

$$\frac{1ly}{1AU} = \frac{9.46 \times 10^{15}}{1.496 \times 10^{11}} = 6.32 \times 10^4$$

$\therefore$ $1\,ly = 6.32 \times 10^4\,AU$

$$\frac{1pc}{1ly} = \frac{3.08 \times 10^{16}}{9.46 \times 10^{15}} = 3.26$$

$\therefore$ $1pc = 3.26\,ly$

$$\frac{1pc}{1AU} = \frac{3.08 \times 10^{16}}{1.496 \times 10^{11}} = 2.05 \times 10^5$$

$\therefore$ $1pc = 2.05 \times 10^5\,AU$

$\therefore$ $1\,pc > 1\,ly > 1\,AU$

The parallax method is used only to measure distances up to 100 pc. This corresponds to a parallax of 0.01 arc second. This is because the angles smaller than 0.01 arc second are very difficult to measure from the earth due to the effects of the atmosphere. This limits the distance measured to about 100 pc. However, the satellite Hipparcos launched in 1989 was able to measure parallax angles to 0.001 arc seconds which corresponds to a distance of 1000 pc.

All known stars have a parallax angle smaller than 1 arc second. For example our second nearest star proxima centuari belongs to centaurus constellation is at a distance of 1.3 pc (4.22 ly) and corresponding parallax is $0.772''$. Another example is the star sirus which belongs α canis Majoris constellation is at a distance of 2.64 pc. (8.6 ly) which corresponds to a parallax of 0.379 arc seconds.

Parallax of some bright and near by stars and the corresponding constellation arc given below.

Note: Hipparcos is the abbreviation of high precision parallax collecting satellite and at the same time remembering great Greek astronomer Hipparchus.

Table 3.1: Nearest stars in the sky

No	Star	Parallax in arc second	Distance in pc	Constellation
1	Proxima centauri	0.772	1.3	Centarus
2	Alpha centauri A	0.741	1.35	Centarus
3	Barnards star	0.55	1.82	Ophichus
4	Wolf 359	0.418	2.39	Leo
5	Lalande 385	0.392	2.55	Ursa Major
6	Sirius A	0.379	2.64	Canis Major
7	UV ceti A	0.374	2.67	Cetus
8	Ross 154	0.336	2.97	Sagittarius
9	Ross 248	0.316	3.16	Andromeda
10	Epsilon Eridani	0.311	3.22	Eradinus
11	HD 217989	0.304	3.29	Piscis Austrinus
12	Ross 128	0.299	3.34	Virgo
13	L789-6A	0.291	3.44	Acquaris
14	61 cygni A	0.287	3.48	Cygnus
15	Procyon A	0.286	3.50	Canis Minoris
16	61 cygni B	0.285	3.51	Cygnus
17	HD 173740	0.284	3.52	Draco
18	HD 173739	0.280	3.57	Draco
19	GX Andromeda	0.280	3.57	Andromeda

We found that distances of stars up to 1000 pc $(3.08\times10^{17}\,\text{km})$ could be determined. But these are all relatively close stars. Most of the stars in the galaxy are too far away for parallax measurements to be taken. So we discuss some other methods of distance measurements in astronomy.

This method is based on the regular variation of brightness (or luminosity) of stars. The stars whose brightness vary at regular intervals of time is called variable stars or pulsating stars (pulsars).

There are two types of variable stars particularly useful in determining distances. These are Cepheid variable stars and RR Lyrae variable stars. These stars change their diameters over a period of time. This period of time can be measured very accurately. Using Period-luminosity relationship we can calculate the luminosity. By comparing the luminosity, which is a measure of the intrinsic brightness of the star with the brightness it appear to have in the sky, its distance can be calculated from the luminosity-brightness relation which contains the distance.

Using Cepheid star as the reference, distances about 18.4 millions parsecs have been determined. Using RR Lyrae stars as the reference, distances about 0.61 million parsec have been determined.

Another method of distance determination is that of spectroscopic parallax. In this method we take the spectra of stars.

From the spectral classification, we will be able to calculate their intrinsic luminosity. Then it will be compared with its apparent brightness to determine its distance.

There are other distance determination methods for the objects farthest away from us such as other galaxies. They are Tully Fisher method, Hubble law method etc.

We conclude this section by saying that all these methods do not produce exact results. The error associated is in between 10% to 25%. Sometimes it may go upto 50%. For example distance of a star is measured 10000 pc. Suppose it has 25% error. This means that star can be found any where between $10000 \pm 10000 \times \frac{25}{100}$ i.e., in between 7500 pc and 12500pc (in between 2.3×10^{20}m and 3.85×10^{20}m).

Brightness and luminosity

There are millions and millions of stars in the sky. Out of which the total number of stars visible to the naked eye in the whole sky is only about 5000. Others can be seen through telescopes. Most of the stars are powered by the same process that fuels the Sun. This does not mean that they are all alike. Stars differ in their size, mass, luminosity etc. One of the most important characteristics is their luminosity(L). **The luminosity of a star is the total amount of radiant energy emitted per second from the surface of a star.** It is measured in watts (W) or as a multiple of the Suns luminosity denoted by $L_{\odot}$ (a circle with a dot indicates Sun in astronomy). However luminosity of a star cannot be measured directly. This is because what we can measure on earth is the amount of light reaching on earth. So we introduce a term called apparent brightness or simply brightness of a star denoted by b. **The amount of light energy reaching the earth per unit area in unit time is called brightness.**

It is measured in Wm^{-2}. The brightness (b) depends on the luminosity (L) and the distance of the star(d). This is as follows.

The light received per unit area in unit time on earth = b

If the star is at a distance of d from the earth. The star would emit light in all directions covering a total area of $4\pi d^2$.

∴ The total energy that can be received by covering the whole sphere of area

$4\pi d^2$ in unit time $= b \cdot 4\pi d^2$.

This is nothing but the light energy emitted by the star in unit time called luminosity.

i.e. $$L = b4\pi d^2$$

or $$b = \frac{L}{4\pi d^2} \qquad \text{..... (2)}$$

This is the relation between luminosity, brightness and distance.

Astronomers measure star's brightness with light sensitive detectors and this procedure is called photometry.

Example 1

The star Sirius A has luminosity 97.28×10^{26} W, and is at a distance of 8.6 *ly*. Calculate its apparent brightness.

Solution

$L = 97.28 \times 10^{26}$ W, $\quad d = 8.6\ ly = 8.6 \times 9.46 \times 10^{15}$ m

The apparent brightness, $b = \frac{L}{4\pi d^2}$

$$b = \frac{97.28 \times 10^{26}}{4 \times 3.14 \times (8.6 \times 9.46 \times 10^{15})^2}$$

$$b = \frac{97.28 \times 10^{26}}{4 \times 3.14 \times 8.14^2 \times 10^{32}}$$

$$b = \frac{97.28}{4 \times 3.14 \times 66.3} \times 10^{-6}$$

$$b = 1.17 \times 10^{-7}\,\text{Wm}^{-2}$$

Example 2

The luminosity of Sun is 3.83×10^{26} W. What is its apparent brightness.

Solution

$$L = 3.83 \times 10^{26}\,W \qquad d = 1\,AU = 1.496 \times 10^{11}\,m$$

Using $b = \dfrac{L}{4\pi d^2} = \dfrac{3.83 \times 10^{26}}{4 \times 3.14 \times 1.496^2 \times 10^{22}}$

$$b = 1363\ Wm^{-2}$$

Example 3

The star aldebaran which belongs to Taurus is at a distance of $65.11\,ly$ has luminosity $518L_\odot$. Calculate its brightness.

Solution

$$d = 65.11\ ly = 65.11 \times 9.46 \times 10^{15}\,m = 6.16 \times 10^{17}\,m$$

$$L = 518L_\odot = 518 \times 3.83 \times 10^{26}\,W$$

$$L = 1.98 \times 10^{29}\,W$$

Using $b = \dfrac{L}{4\pi d^2}$

$$b = \frac{1.98 \times 10^{29}}{4 \times 3.14 \times 6.16^2 \times 10^{34}}$$

$$b = 4.15 \times 10^{-6}\,Wm^{-2}$$

Luminosity, brightness and distance of stars in terms of that of Sun's.

We have $L = 4\pi d^2 b$

For the Sun $L_\odot = 4\pi d_\odot^2 b_\odot$

$$\therefore \qquad \frac{L}{L_\odot} = \left(\frac{d}{d_\odot}\right)^2 \left(\frac{b}{b_\odot}\right) \qquad \text{..... (3)}$$

Here what we require is the distance of star measured in terms of earth-sun distance and brightness of star measured in terms of brightness of the sun with respect to earth. So we get the luminosity of the star in terms of $L_\odot$.

Example 4

Consider two stars A and B. Star A be at a distance half that of B and appear twice as bright as B. Compare their luminosities.

Solution

$$d_A = \frac{d_B}{2}, \quad b_A = 2b_B$$

Using $L = \pi d^2 b$

$$\therefore \quad \frac{L_A}{L_B} = \frac{d_A^2 b_A}{d_B^2 b_B} = \left(\frac{d_A}{d_B}\right)^2 \cdot \left(\frac{b_A}{b_B}\right)$$

$$\frac{L_A}{L_B} = \left(\frac{1}{2}\right)^2 \cdot 2 = \frac{1}{2}$$

$$L_A = \frac{L_B}{2} \quad \text{or} \quad L_B = 2L_A$$

Though star B is twice luminous than A, it appears to be less brighter.

Example 5

Consider two stars A and B. Star A appears to be half as bright as B and distance of A is two times that of B. How much is the luminosity of star A compared with respect to star B.

Solution

$$b_A = \frac{1}{2} b_B, \quad d_A = 2d_B$$

Using $L = \pi d^2 b$

$$\frac{L_A}{L_B} = \frac{d_A^2}{d_B^2} \frac{b_A}{b_B} = \left(\frac{d_A}{d_B}\right)^2 \cdot \left(\frac{b_A}{b_B}\right) = 2^2 \cdot \frac{1}{2} = 2$$

$$\therefore \quad L_A = 2L_B$$

Magnitudes of stars

We could see that the apparent brightness of a star differs from its luminosity.

The apparent brightness depends on its luminosity as well as on the distance of star from us. Based upon the measurements of apparent brightness of stars, a measurement scale had been developed called magnitude. It is denoted by m. During the second century B.C, Hipparchus (Greek astronomer) classified the naked eye stars into six magnitudes according to their apparent brightness. He catalogued nearly 1000 stars. The brightest star is assigned magnitude one (m = 1). The star of half as bright as first magnitude is called as second magnitude star (m = 2) and so on to the sixth magnitude star which are faintest star in the sky. In other words we can say that the first magnitude star is two times brighter than second magnitude star. The second magnitude star is two times brighter than third magnitude star. So the first magnitude star is 2^2 times brighter than third magnitude star. As there are six magnitudes, we can say that first magnitude star is 2^5 times brighter than sixth magnitude star.

In about 1830 William Herschel by his experiment on Stellar photometry determined that stars in one magnitude class are 2.512 times more brighter than the stars in the next higher magnitude class.

i.e. $$\frac{\text{Brightness of first magnitude star}}{\text{Brightness of second magnitude star}} = (2.512)^{2-1}$$

$\therefore$ $$\frac{\text{Brightness of first magnitude star}}{\text{Brightness of sixth magnitude star}} = (2.512)^{6-1} = (2.512)^5 = 100$$

Let b_1 be the brightness of first magnitude star and b_6 be the brightness of sixth magnitude star, the above relation can be written as

$$\frac{b_1}{b_6} = (2.512)^5 = (2.512)^{6-1} = (2.512)^{m_6 - m_1}$$

or in general $$\frac{b_1}{b_2} = (2.512)^{m_2 - m_1}$$

i.e. $$\frac{b_1}{b_2} = (100)^{\frac{m_2 - m_1}{5}} \qquad \left(\because \quad (100)^{\frac{1}{5}} = 2.512\right)$$

Taking log on both sides, we get

$$\log \frac{b_1}{b_2} = \frac{(m_2 - m_1)}{5} \log 100 = \frac{(m_2 - m_1)}{5} \times 2$$

$$\therefore \qquad m_2 - m_1 = 2.5 \log \frac{b_1}{b_2}$$

or $$m_2 - m_1 = -2.5 \log \frac{b_2}{b_1} \qquad ..... (4)$$

This is the relation between apparent brightness and magnitude.

Another form of the relation between apparent brightness and magnitude

We have $$\frac{b_1}{b_2} = (2.512)^{m_2 - m_1}$$

Taking log on both sides, we get

$$\log\left(\frac{b_1}{b_2}\right) = (m_2 - m_1)\log 2.512$$

$$\log\left(\frac{b_1}{b_2}\right) = (m_2 - m_1) \times 0.4$$

$$\therefore \qquad \frac{b_1}{b_2} = 10^{0.4(m_2 - m_1)} \qquad ..... (5)$$

If $b_1 = b_0$, the apparent brightness of zero magnitude star i.e $m_1 = 0$. From equation 4, we get

$$m_2 = -2.5 \log \frac{b_2}{b_0}$$

In general $$m = -2.5 \log \frac{b}{b_0} \qquad ..... (6)$$

The star Vega (Abhijith) is of zero magnitude and it's brightness measured is $b_0 = 2.52 \times 10^{-8}\,Wm^{-2}$

The apparent magnitudes of some stars and planets are given below.

Table 3.2: Some objects in the sky and their magnitudes

No	Object	Magnitude (m)
1	Sun	–26.7
2	Full moon	–12.6
3	Venus	–4.4*
4	Jupiter	–2.6*
5	Sirius A	–1.44
6	Sirius B	+8.66
7	Faintest observable star	+23

Note: * When brightest

In the table 3 given below magnitude difference and brightness ratio are shown.

Table 3.3: Magnitude difference and brightness ratio

Magnitude difference $m_2 - m_1$	Brightness ratio $\frac{b_1}{b_2} = 10^{0.4(m_2 - m_1)}$
0.0	1.0
0.1	1.1
0.2	1.2
0.3	1.3
0.4	1.45
0.5	1.58
0.6	1.74
0.7	1.91
1	2.51(2)
2	6.31
3	15.85
4	39.8
10	10000
15	1000000
20	100000000

Example 6

The Sun has a magnitude of –26.7 and the full moon has a magnitude of –12.6. Find the apparent brightness of Sun compared to that of full moon.

Solution

$$m_\odot = -26.7$$

$$m_{moon} = -12.6$$

Using $m_2 - m_1 = 2.5 \log \frac{b_1}{b_2}$

We have $m_{moon} - m_\odot = 2.5 \log \frac{b_\odot}{b_{moon}}$

$$-26.7 - -12.6 = 2.5 \log \frac{b_\odot}{b_{moon}}$$

$$\frac{-14.1}{2.5} = \log \frac{b_\odot}{b_{moon}}$$

$$\log \frac{b_\odot}{b_{moon}} = -5.64$$

or $$\frac{b_\odot}{b_{moon}} = 10^{-5.64} = 2.29 \times 10^{-6} = \frac{1}{4.37 \times 10^5}$$

$\therefore$ $$b_\odot = \frac{b_{moon}}{4.37 \times 10^5}$$

This shows that, though the luminosity of sun is greater than twice the luminosity of moon, its brightness is about 1 millionth of that of moon.

Example 7

Compare the apparent brightness of Sun and the star Sirius A. $m_\odot = -26.7$ and $m_{sirius} = -1.44$

Solution

$$m_{\odot} = -26.7\,, \quad m_{sirius} = -1.44$$

We have $m_2 - m_1 = 2.5\log\frac{b_1}{b_2}$

$$m_{sirius} - m_{\odot} = 2.5\log\frac{b_{\odot}}{b_{sirius}}$$

$$-1.44 - -26.7 = 2.5\ \log\ \frac{b_{\odot}}{b_{sirius}}$$

$$\frac{25.26}{2.5} = \log\ \frac{b_{\odot}}{b_{sirius}}$$

$$10.104 = \log\ \frac{b_{\odot}}{b_{sirius}}$$

$$\therefore \qquad \frac{b_{\odot}}{b_{sirius}} = 10^{10.1} = 1.259 \times 10^{10}$$

This problem shows that even though Sirius is much luminous than sun ($L_{sirius} = 25.4L_{\odot}$), it apparent brightness is $\frac{1}{1.259\times10^{10}}$ times fainter than the sun.

Absolute magnitude (M)

The apparent magnitude of a star does not tell us whether a star is actually bright or not. It tells us only about the apparent brightness. The apparent brightness of a star depends on the luminosity and the distance of the star. Of two stars are at the same distance their brightness are proportional to their luminosities. As stars are at different distances, their apparent brightness do not provide their luminosities. If we know the actual distances of stars, we can compare their luminosities by referring them to any chosen distance from us. For such a comparison of luminosities of stars, astronomers have chosen a standard distance of 10 parsecs (32.6 light years). The magnitude of the star is then called the absolute magnitude (M). **Thus the absolute magnitude (M) of a star has been defined as its apparent brightness if it were placed at a standard distance of 10 parsecs.** The absolute magnitude of a star, therefore depends on its actual brightness not on its actual distance.

Relation between apparent magnitude, absolute magnitude

Consider a star whose apparent brightness is b_1 and apparent magnitude m at a distance d. If the same star is at a distance D, let its apparent brightness be b_2 and apparent magnitude M.

From equation 5, we have

$$\frac{b_1}{b_2} = 10^{0.4(M-m)} \qquad \text{..... (7)}$$

From equation (2), the apparent brightness is

$$b = \frac{L}{\pi d^2}$$

Thus $$b_1 = \frac{L}{\pi d^2}$$

and $$b_2 = \frac{L}{\pi D^2}$$

Luminosity is same since only one star.

$$\therefore \quad \frac{b_1}{b_2} = \frac{D^2}{d^2}$$

Put this in equation (7), we get

$$\frac{D^2}{d^2} = 10^{0.4(M-m)}$$

Taking log on both sides we get

$$2(\log D - \log d) = 0.4(M - m)$$

$$\therefore \quad M - m = 5(\log D - \log d)$$

When $D = 10\text{pc}$, M is called the absolute magnitude

$$M - m = 5(\log 10 - \log d) = 5(1 - \log d)$$

or $$m - M = 5(\log d - 1) \qquad \text{..... (8)}$$

This is the relation between apparent magnitude m, absolute magnitude M and distance d.

The quantity m – M, depends only on d, is called the distance modulus. Knowing the distance modulus the actual distance of star can be calculated from equation 8.

The absolute magnitude of the normal stars lie between –10 to +20. The range corresponds to a brightness ratio of about 10^{12}.

Example 8

The star Sirius A is at a distance of 2.63 pc and has an apparent magnitude of–1.44. Calculate its absolute magnitude.

Solution

$d = 2.63\text{pc}, \; m = -1.44$

Using $$m - M = 5(\log d - 1)$$

$$\therefore \quad M = m - 5(\log d - 1)$$

$$M = -1.44 - 5(\log 2.63 - 1)$$

$$M = -1.44 - 5(0.42 - 1)$$

$$M = -1.44 - 5 \times -0.58$$

$$M = -1.44 + 2.9$$

$$M = -1.46$$

Example 9

The Sun has an apparent magnitude of -26.7. Calculate its absolute magnitude

Solution

$$m = -26.7 \quad d = 1\text{AU} = \frac{1.496 \times 10^{11}}{3.08 \times 10^{6}}\text{pc} = 4.86 \times 10^{-6}\,\text{pc}$$

Using $$m - M = 5(\log d - 1)$$

$$\therefore \quad M = m - 5(\log d - 1)$$

$$M = -26.7 - 5(\log 4.86 \times 10^{-6} - 1)$$

$$M = -26.7 - 5 \times (-5.31 - 1)$$

$$M = -26.7 + 31.6$$

$$M = 4.9$$

Example 10

The star Vega belongs to Lyra constellation has an apparent magnitude zero is at a distance of 25.3 *ly*. Calculate its absolute magnitude.

Solution

$$m = 0,\ d = 25.3ly = \frac{25.3}{3.26}pc = 7.76pc$$

Using $$m - M = 5(\log d - 1)$$

$\therefore$ $$M = m - 5(\log d - 1)$$

$$M = 0 - 5(\log 7.76 - 1)$$

$$M = -5 \times 0.11 = 0.55$$

Example 11

The Betelgeuse belongs to Orion constellation has an apparent magnitude of 0.45 and absolute magnitude -5.14. Calculate its distance in light years.

Solution

$$m = 0.45, \quad M = -5.14$$

Using $$m - M = 5(\log d - 1)$$

$$0.45 - -5.14 - 5(\log d - 1)$$

$$\frac{5.59}{5} = \log d - 1$$

$\therefore$ $$\log d = 1 + \frac{5.59}{5} = 1 + 1.118 = 7.118$$

$$d = 10^{2.118} = 131.22pc$$

$$d = 131.22 \times 3.26\ ly$$

$$d = 427.7ly$$

Colour and temperature of stars

When we look up into the clear sky at night with naked eye we see several stars. All of them are generally white. This colour varies from person to person. This is because eye does not recognize colour at low light levels. This why at night with the naked eye we see only shades of grey, white, pale yellow etc. Moreover the most important factor determining the colour of a star you see is you-the observer. It is purely a matter of physiological and psychological influences. What one observer describes as a blue star another may describe as a white star or one may see an orange star, while another observes the same star as yellow. You might even observe a star to have different colour when using different telescopes or magnifications and atmospheric conditions certainly have a role to play. But there are some exceptions to these. A few stars exhibit distinct colours. For example Betelgeuse (α Orionis) and Antare (α Scorpi) are definitely red, Capella (α Aurigae) is yellow and Vega (α Lyrae) is steely blue. However in the case of naked eye stars, there is no much variation in colour. The scenario will be different when you look at stars with telescope, there seems to be much variations in colours.

The colour of a star is determined by its surface temperature. A red star has a lower temperature than that of a yellow star, which in turn has a lower temperature than that of a blue star etc. **The relation between colour and temperature of a star is explained by Wien's displacement law. According to the law**

$$\lambda_m T = \text{Constant} = 2{,}900{,}000 \text{ nmK} \qquad \text{..... (9)}$$

where λ_m is the wavelength corresponding to maximum intensity and T is the temperature. This law states that low temperature stars emit most of their energy in the red to infrared part of the spectrum, while much hotter stars emit in the blue to ultraviolet part of the spectrum. When stars emit energy in the ultraviolet or infrared region we cannot see them. The low temperature stars make up about 70% of the stars in our galaxy but you would never ever see them.

An important point to notice here is that hotter objects emit more energy ($\varepsilon \propto T^4$) at all wavelengths due to the higher average energy of all the photons. The three graphs shown below demonstrate how the light from three different stars is distributed depending on the star's temperature.

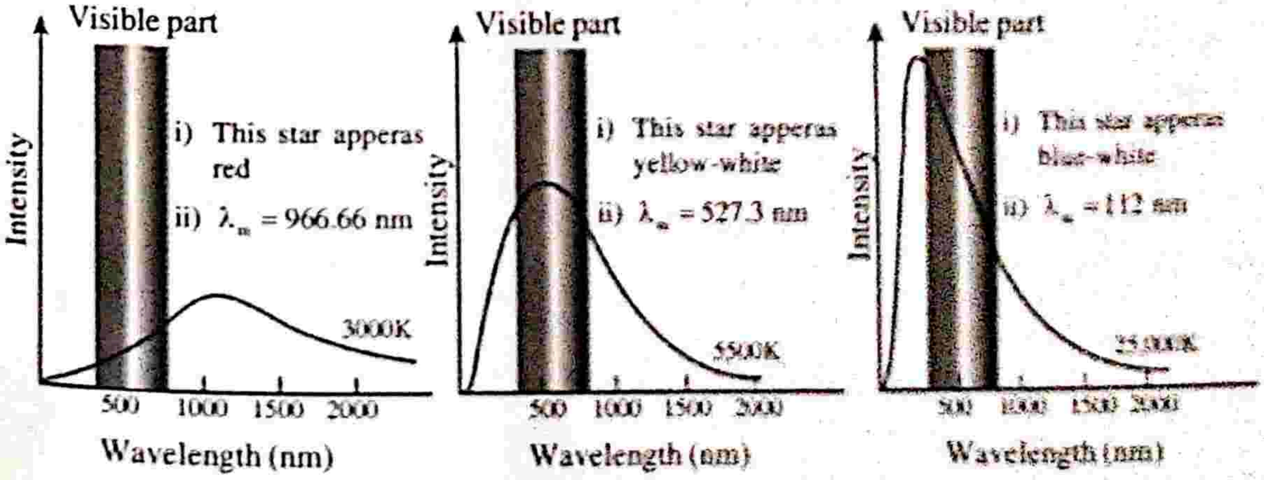


Figure 3.2: Relation between colour and temperature

Another interesting thing to observe is that a few stars are so hot, possibly in millions of degrees, emit X-rays. These are neutron stars.

Note: When we say about star's temperature it refers to its surface temperature.

Temperature of stars not only enables us to determine its colour but also helps to determine many other characteristics. One scientific description of a star's colour is based on the Stellar classification. Another term commonly used by astronomers in association with star's colour is colour index.

Example 12

Two stars α Canis Majoris and 0 Ceti, have a temperature of 9200K and 1900K respectively. What are their peak wavelengths?

Solution

$$T = 9200K \text{ for } \alpha \text{ Canis Majoris}$$

$$\text{We have } \lambda_{max} = \frac{2,900,000}{T} \text{nm}$$

$$\lambda_{max} = \frac{2,900,000}{9200} = 315\text{nm}$$

This wavelength is in the ultraviolet region.

$$T = 1900K \text{ for 0 Ceti}$$

$$\lambda_{max} = \frac{2900000}{1900} = 1526\text{nm}$$

This wavelength is in the infrared region.

Example 13

The mysterious star Zubeneschamali belongs to Libra constellation has a temperature of 11,000K. What is its peak wavelength?

Solution

$$T = 11,000K$$

$$\text{We have} \quad \lambda_n = \frac{2,900000}{T} \text{nm}$$

$$\lambda_n = \frac{2,900000}{11,000} = 263.6\text{nm}$$

It is one of the rare green coloured stars.

Size and mass of stars

When we look at stars they seem to be like point light sources since they are several light years away from us. So how do we determine the size of a star is our concern now. Knowing the luminosity (L) and surface temperature (T) of a star we can easily find out the size of a star. Luminosity of a star is determined by knowing apparent brightness (b) and its distance d using the formula $L = 4\pi d^2 b$. The temperature of a star can be calculated from the spectra given out by the star using $\lambda_m T = 0.0029\,mK$ knowing L and T, how can we determine the size of a star. For this we make use of Stefan's law. According to this law the energy emitted per unit area in one second is proportional to the fourth power of the temperature T. The energy emitted per unit area in unit time is called energy flux (F). Thus according to Stefan's law

$$F \propto T^4$$

or

$$F = \sigma T^4 \qquad \text{..... (10)}$$

The constant of proportionality σ is called Stefan's constant and its value is given by

$$\sigma = 5.67 \times 10^{-8}\,Wm^{-2}K^{-4}$$

Relation between flux, luminosity and radius

Consider a star spherical in shape with a radius R. Its surface area is $4\pi R^2$. From the definition of luminosity, we have

$$L = \frac{\text{Energy emitted}}{\text{time}}$$

or

$$L = \frac{\text{Energy emitted} \times \text{Area}}{\text{Area} \times \text{time}}$$

By definition $\frac{\text{Energy emitted}}{\text{Area} \times \text{time}} = \text{Energy flux (F)}$

$$\therefore \quad L = F \times \text{Area} = F \cdot 4\pi R^2 \qquad \text{..... (11)}$$

This is the relation between L, F and R. Substituting for F

$$L = \sigma T^4 4\pi R^2$$

i.e.

$$L = 4\pi R^2 \sigma T^4 \qquad \text{..... (12)}$$

Knowing L, σ and T, the size of the star can be calculated.

Note: Most of the stars are spherical in shape, a few are not.

The equations 10 and 12 tell us that, a cold star has low flux (eqn. 10), but it may have large luminosity if radius is large (eqn. 12). Similarly a hot star will have low luminosity it has a small radius. This implies that temperature alone cannot indicate how luminous a star will be, the radius is also needed.

Suppose we want to compare the size (radius) of a star with reference to size of the sun.

We have $L = 4\pi R^2 \sigma T^4$

$$\therefore \quad L_\odot = 4\pi R_\odot^2 \sigma T_\odot^4$$

Thus $$\frac{L}{L_\odot} = \left(\frac{R}{R_\odot}\right)^2 \left(\frac{T}{T_\odot}\right)^4$$

or $$\frac{R}{R_\odot} = \left(\frac{L}{L_\odot}\right)^{\frac{1}{2}} \left(\frac{T_\odot}{T}\right)^2 \qquad \text{..... (13)}$$

Example 14

The star aldebaran belongs to taurus constellation has a temperature of about 3910K and luminosity of about 518 $L_\odot$. What is its size with reference to that of the Sun? $T_\odot = 5800K$.

Solution

$T = 3910K \qquad L = 518L_\odot$

We have $$\frac{R}{R_\odot} = \left(\frac{T_\odot}{T}\right)^2 \left(\frac{L}{L_\odot}\right)^{\frac{1}{2}}$$

$$\frac{R}{R_\odot} = \left(\frac{5800}{3910}\right)^2 (518)^{\frac{1}{2}} = 2.2 \times 22.76$$

$$\therefore \quad R = 50.1\ R_\odot$$

In table 4 given below some giant and supergiant naked eye stars with size, luminosity and distance are shown

Table 3.4

No	Star	Radius in AU	Distance in ly	Luminosity in $L_\odot$
1	α -Herculus	2	400	7244 - 9333
2	ψ1 -Aurigae	2.88	4300	11,000
3	η -Persei	2	1300	4000
4	VV Cephei	8.8	2000	2×10^5
5	KQ Puppis	8.8	3361	>9870

Stellar constituents

Here we discuss about what stars are made of. A star is an enormous sphere of hot gases. The gas is composed of hydrogen, helium and some other elements (metals). Astronomers call every element other than hydrogen and helium a metal. The composition is usually about 75% of hydrogen, 24% helium and the remainder metals. This ratio may change, however very old stars are nearly all hydrogen and helium with tiny amount of metals and very few stars can contain 2-3 % metals.

The energy needed to create and maintain a star is produced within the star by nuclear fusion. For this a very high temperature and a strong gravitational fields are required. It is due to very large mass, strong gravitational fields are present. The temperature at the centre of star may be about 10 million kelvin. This results in huge pressure at the centre of the star. This huge pressure and enormous heat trigger nuclear reaction at the centre of the star. This nuclear reaction converts hydrogen into helium releasing small amount of energy when billions of such reactions takes place producing substational amount of energy, which makes a star shine.

This nuclear reaction continues for millions and million of years till all hydrogen are exhausted. The by-product of this nuclear reaction is helium when helium present is in abundance and under suitable conditions (higher temperature and a large mass), helium undergoes nuclear reactions at the core of the star. After a very long time the by-product this reaction which will be in abundance. If conditions are favourable carbon too will initiate nuclear fusion and the process continues. At each stage the temperature required is much greater than the previous stage. If this condition is not satisfied, at any stage further reaction will not occur. This shows that burning of hydrogen and helium are the source of power for nearly all the stars that we see. The mass of the star will determine how the reaction will proceed.

Stellar spectra

The spectrum given out by a star is called stellar spectrum. The stellar spectra plays an important role in astrophysics. This is because by analysing the spectra we can determine several parameters of the star such as its temperature, colour, distance, in which direction it is moving, whether it is rotating or not. In addition to these we can infer its age, mass, how long will it live and so on. Another important aspect is the spectral classification of stars.

To get solar spectrum what we need is a spectroscope, a prism or a diffraction grating a telescope and an eye piece. Using the spectroscope mounted at the eye piece end of a telescope, light from a star can be collected and photographed. Usually star emits different colours. The prism or diffraction grating helps us to disperse different colours. The result we obtain is called a spectrum.

The spectrum contains emission lines as well as absorption lines. Emission lines are due to emission of photon when particles jump from excited states to lower states. When the light from inner part of star passes through outer most cooler layer known as reversing layer some wavelengths are absorbed, depending upon the elements present in the reversing layer give rise to absorption lines. Absorption lines are indications of the elements present in the star. The factor that determines whether an absorption line will arise is the temperature of star's atmosphere. A hot star will have different absorption lines than a cool star. By examining the spectrum and measuring various aspects of the absorption lines the classification of a star is determined. Analysis of the structure of absorption lines gives information regarding the pressure, rotation and even the presence of a companion star.

Stellar classification

We found that absorption lines were present in the spectra of all the stars, when the absorption spectra (which depends on temperature and indicating the presence of metals) of stars were studied, it was realised that stars could be classified into several different types called spectral classes. This classification was firstly done by Harward group of astrophysicists under the direction of E.C. Pickering and his associates. This is called Harward classification or Pickering classification. It is also called Henry Draper catalogue after the name of Henry Draper. Harward people classified the spectra of about 4,00,000 stars. It was later realised that the types of spectra varied primarily because of differing temperatures of the stellar atmospheres. Much before this the Indian astrophysicist M.N. Saha suggested that the difference in stellar spectra are principally due to surface temperatures of stars. It means that the spectral classes correspond in fact to different surface temperatures. Harward classification is done in the order of decreasing temperature. The classification, is designated by capital letters, is written as

O B A F G K M R N S H

This can be remembered by the mnemonic. Oh! Be A Fine Girl Kiss Me Right Now Sweet Heart.

The spectral types are further divided in ten subclasses beginning with 0 and ending at 9.

Among the group O group stars are hottest and H group stars are coolest. Further class A1 star is hotter than class A9 star. But A9 star is hotter than F0 star. The spectra of the hotter star of types O, B, A are sometimes referred to as early type stars while the cooler ones K, M, R, N, S, H are later type. F and G are designated intermediate type star. Our sun is a G2 star, thus it is an intermediate star.

It is interesting to note the distribution of stars throughout the galaxy. A casual glance at the stars in the night sky will give you several O and B type, a few A type, some F and G type, a smattering of K and more M types. A vast majority of stars in our galaxy over 72% of them are faint cool and red M type stars. The bright and hot O-type stars are less than 0.005%. For every O type there are about 1.7 million M-types. Now we shall briefly discuss how the spectrum are affected by temperature. We found that star is composed of 75% of hydrogen. Hydrogen emits various spectral lines such as Lyman, Balmer, Paschen, Brackett and Pfund series. For simplicity we take Balmer series for explanation. Balmer series absorption lines are obtained when electrons undergo transition from n = 2 level to higher levels. However Balmer lines do not always appear in a stars spectrum. This happen when the temperature is greater than 10,000K. At this temperature photons coming from the interior of stars have high energy that they can easily knock out electrons from hydrogen atoms. i.e, ionisation takes place. In the ionised state it cannot produce Balmer lines. Type O stars up to type B_2 exhibit no Balmer lines.

When the temperature is less than 10,000K most of the hydrogen atoms are in the groundstate. Many of the photons passing through the stars atmosphere do not have enough energy to excite electrons from ground state to energy level 2. Thus there is no Lymann absorption series in the spectra. However very few atoms in the energy level 2 can absorb the photons characteristic energy and excited to higher levels. so absorption lines are found in the spectra. Cool stars M0 and M2 are examples for this.

The principle features of spectral sequence are given in table 5 below.

Note: The Sun - A G2 star has a spectrum dominated by lines of calcium and iron.

Table 3.5: The principal features of the Harward Spectral Sequence

Spectral (star) type	Colour	Temperature	Average Mass (The Sun = 1)	Average Radius (The Sun = 1)	Average Luminosity (The Sun = 1)	Main characteristics	Examples
O	Blue to bluish white	50000 K to 30000 K	60	15	1,400,000	Lines of singly ionized helium either in absorption or emission	Naos Mintaka 10 Lacertra
B	Blue to Bluish white	30000 K to 10000 K	18	7	20,000	Lines of neutral helium in absorption	Rigel spectra (Orion)
A	White	10000 K to 8000 K	32	2.5	80	Hydrogen (H) lines are very strong for AO stars, decreasing for A stars	Vega and Sirius
F	White to Yellowish white	8000 K to 6000 K	1.7	1.3	6	Lines of Ca II are strong. Metallic lines such as iron and titanium are also seen	Procyon Canopus
G	Yellowish white to Yellow	6,000 K to 4,500 K	1.1	1.1	1.2	Lines of neutral metallic atoms (Ca II) and ions	Sun, Capella
K	Deep yellow to orange to red	4500 K to 3500 K	0.8	0.9	0.4	Absorption of lines of neutral metals and other elements	Arcturus and Aldebaran
M	Red	3500 K to 2000 K	0.3	0.4	0.04	Molecular bonds of (TiQ dominate the spectrum	Barnard, Betel-geuse and Antares

Hertz sprung-Russell (H-R) diagram

We learned that star's basic characteristics such as its radius, mass, spectral class, absolute magnitude, luminosity and temperature. Now we put all these parameters in a single graph known as Hertz Sprung-Russell diagram. It is one of the most important and useful diagrams in the study of astronomy. The H-R diagram forms the basis of the theory of stellar evolution.

In 1911, the Danish astronomer Ejnar Hertzsprung plotted the absolute magnitude of stars (a measure of their luminosities) on the vertical axis and their colours (a measure of their temperatures) on the horizontal axis. Later in 1913, the American

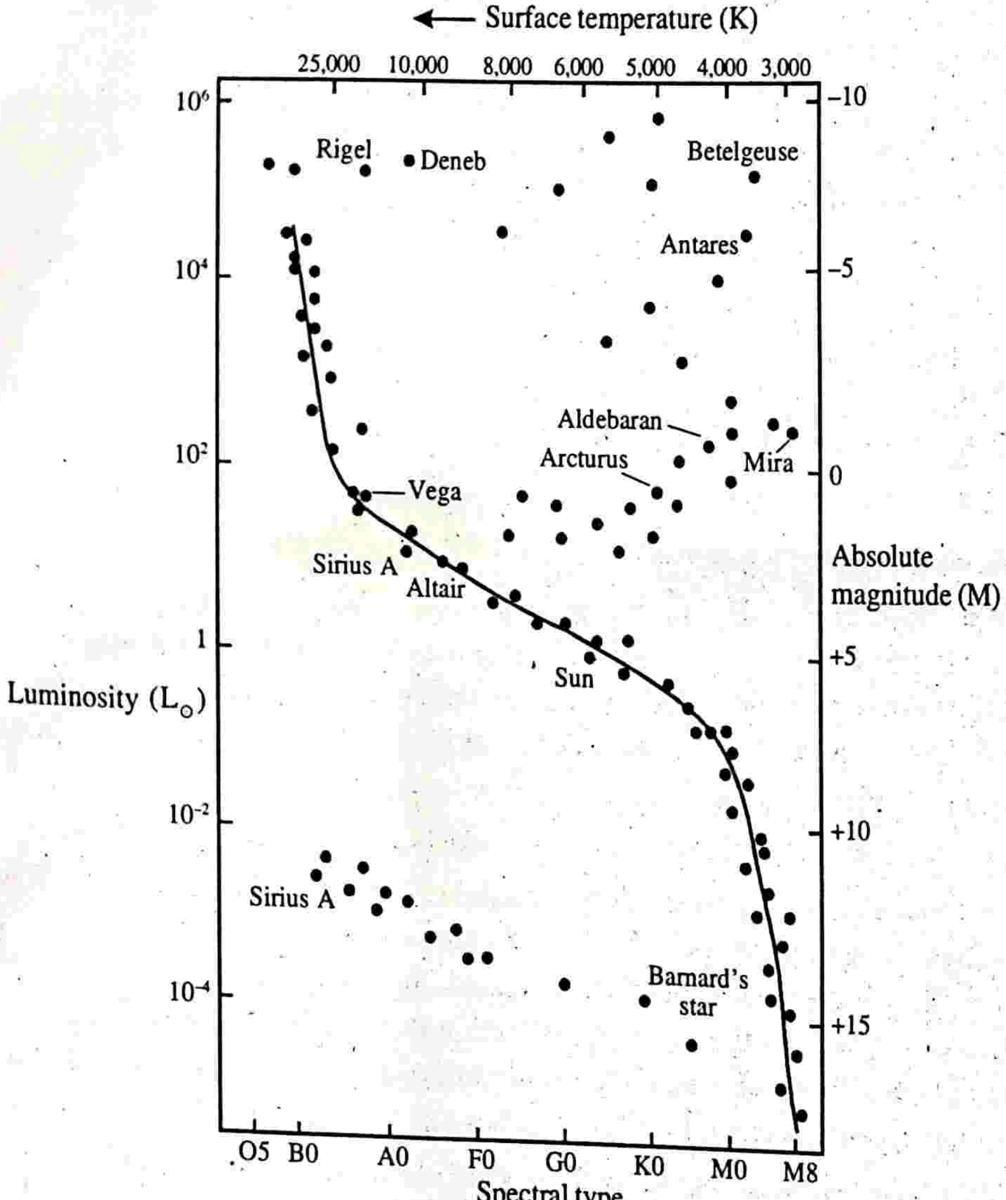


Figure 3.3: Hertzprung-Russel diagram

astronomer Henry Norris Russell independently plotted spectral types (measure temperature) on the horizontal axis and the absolute magnitude on the vertical axis. They both realised that certain unsuspected patterns began to emerge. Further more an understanding of these patterns was crucial to the study of stars. In recognition of the pioneering work of these astronomers, the graph was known as Hertz sprung-Russell diagram or H.R diagram. A plot of luminosity of stars versus their surface temperatures is known as Hertz Sprung-Russell diagram or H-R diagram. A typical H-R diagram is shown below. Each dot on the diagram represents a star whose properties such as spectral type and luminosity can be determined. Note the key features of the diagram.

(i) The temperature increases from right to left. The hot O types are on the left and cool M-types stars are on the right.

(ii) The luminosities cover a wide range, so the diagram makes use of the logarithmic scale, where by each tick mark on the vertical axis represents a luminosity 10 times larger than the prior one.

(iii) Each dot can give spectral type and corresponding luminosity in terms of Sun's luminosity.

It can be seen that stars near the upper left corner are hot and luminous. The stars near the upper right corner are cool and luminous. Stars near the lower right corner are cool and dim. Stars near the left corner are hot and dim.

H-R diagram and stellar radius

H-R diagram provides another important information about the radius of stars. This is because luminosity and temperature are related through $L = 4\pi R^2 \sigma T^4$. Knowing L and T we can very well calculate the radius of stars. A graph between logarithmic luminosity along the vertical axis and temperature along the horizontal axis (temperature in the decreasing order) is plotted we get H-R diagram indicating radii of stars is given below.

From the graph following information can be drawn.

(i) Stars having same radius lie along a straight lines. (dotted diagonal lines). At a constant temperature we can see that radius increases with luminosity. As temperature decreases and radius increases luminosity is found to increase. This shows that stars in the lower most left part of the graph stars having high temperature, low luminosity and small radius, where as stars in the uppermost part having high intensity large radius but low temperature.

(ii) Stars in the lower left of the H-R diagram are much smaller in radius (about $0.01R_{\odot}$) and appear to be white. They are hot stars with low luminosities. They

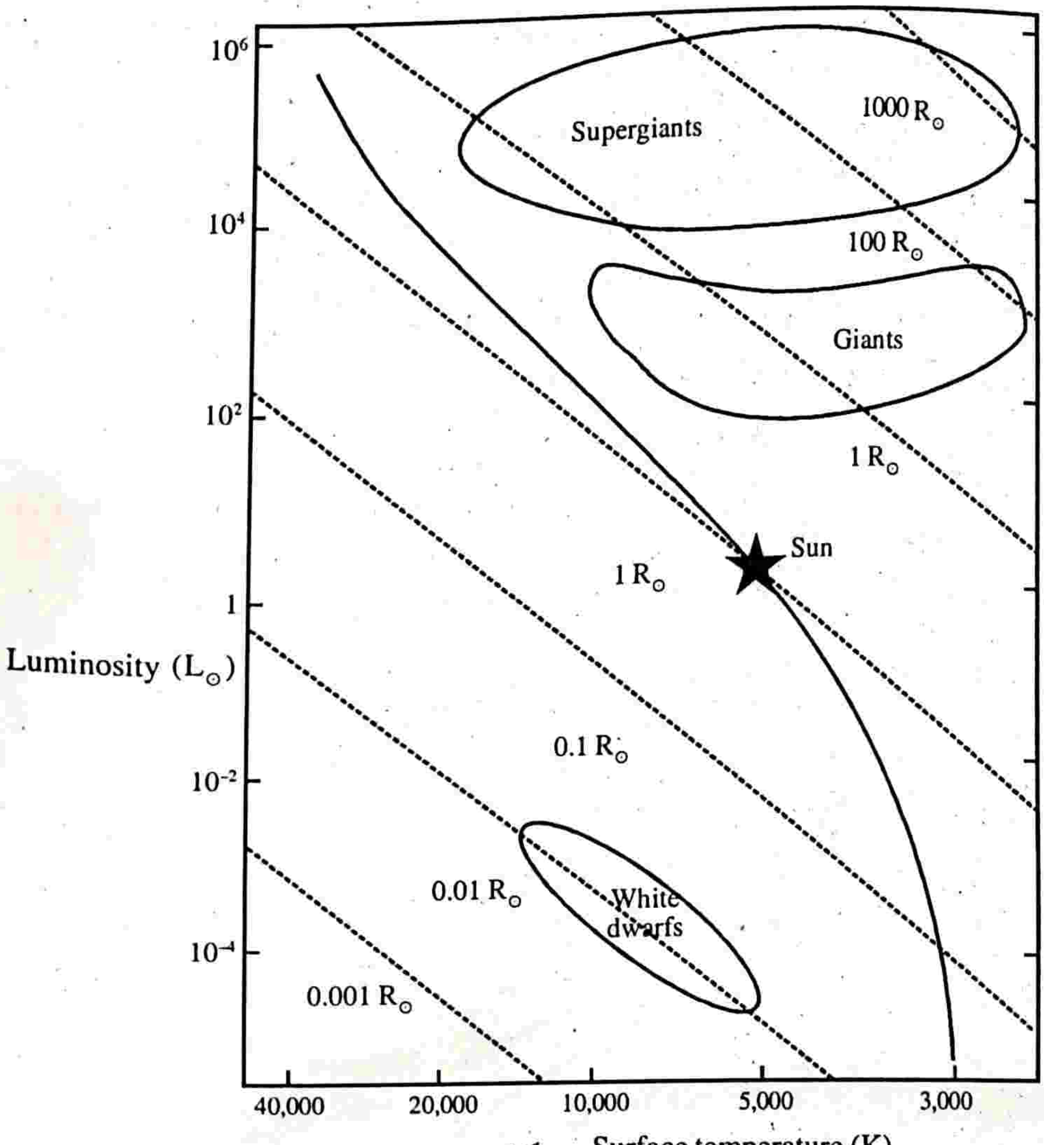


Figure 3.4: Radius of stars on H-R diagram

are called white dwarfs. They are faint stars and can be seen only through telescopes. There are no nuclear reactions in these stars but still glowing. They are remnants of giant stars. White dwarfs have approximately the same size of the earth.

(iii) Stars in the upper right are called giants. They are 10 to 100 times bigger and 100 to 1000 times luminous than the sun. They are cool stars, temperature lies between 3000 to 6000K. Many of the much cooler members of this group are reddish in colour and referred to as red giant. Arcturus in Bootes and aldebaren in taurus are red giants.

(iv) At the extreme right corner there are few stars having radii up to $1000R_\odot$. These are called supergiants. Giant and super giant make up about 1% of stars in the night sky. Antares in scorpius and Betelgeuse in orion are super giants. Their luminosity is very high.

(v) In the diagram you can see a solid line (band) stretches diagonally across it. 90% of stars in the night sky are lying in this band. These are called main sequence. The band along which most of the stars are clustered is called the main sequence. It extends from cool and dim stars in the bottom right to hot and luminous blue stars in the upper left corner. Sun is a main sequence star.

Note: For clarity dots representing stars are not shown in the graph.

H-R Diagram and stellar luminosity

The temperature of a star determines which spectral lines are most prominent in its spectrum. Therefore classifying a star by its spectral type is essentially the same

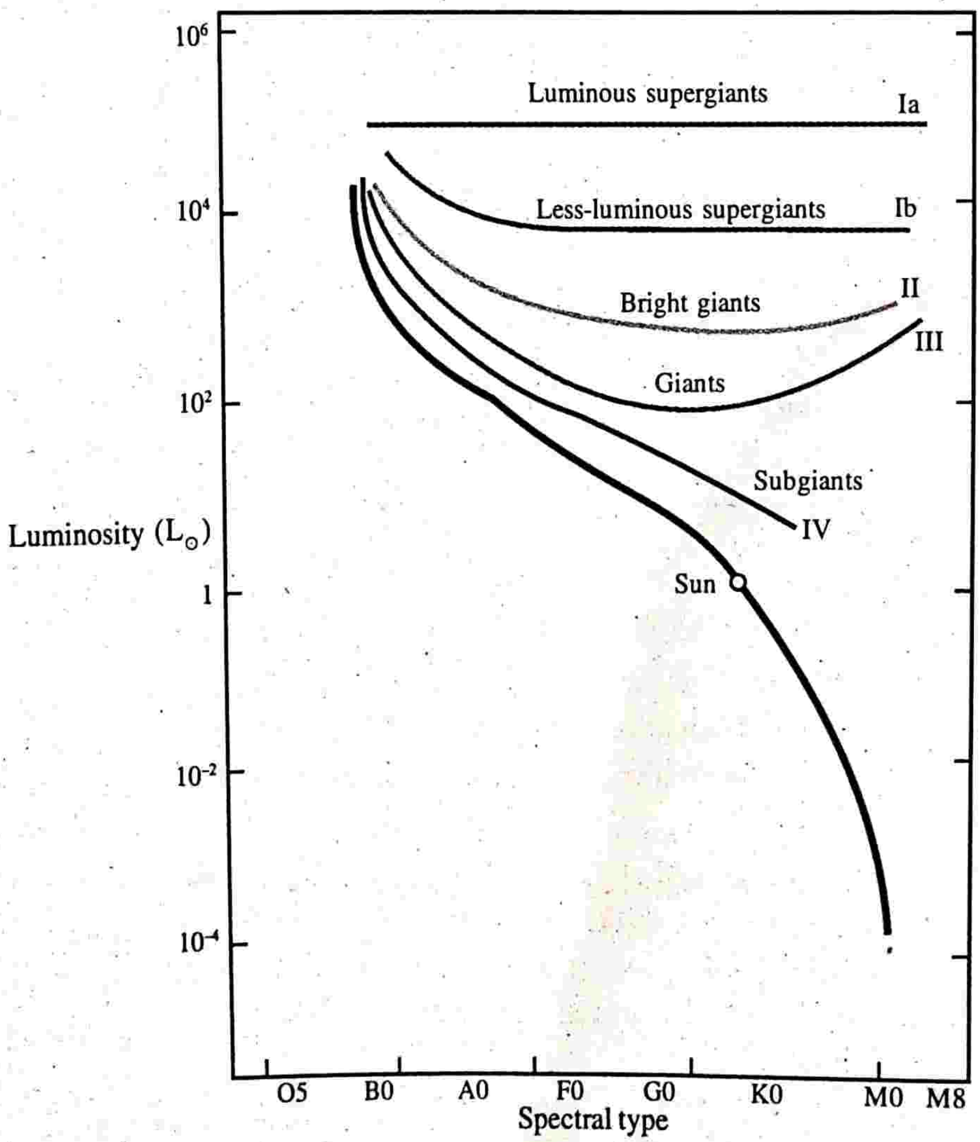


Figure 3.5: Luminosity classes

as by its temperature. The H-R diagram (L versus T) shows that stars can have similar temperatures but with different luminosities.

For example consider a white dwarf star at 7000K. At this temperature drawn perpendicular line we can see that a large number of stars such as main sequence star, a giant, a super giant etc. belong to this temperature but all are having different luminosities. Different luminosities give rise to different spectra. Therefore by examining a star's spectral lines, one can determine which category the star belongs to. Difference in spectral line means difference in the absorption lines of the spectra. It depends upon various conditions on the stars atmosphere. For example the density and pressure of hot gases in the atmosphere affect the absorption lines and hydrogen in particular. If the density and pressure are high, hydrogen atoms collide more frequently and they interact with other atoms in the gas. The collisions can use the energy levels in hydrogen atoms to shift resulting in broadened hydrogen spectral lines.

In a giant luminous star, the atmosphere will have very low pressure and density because the stars mass is spread over such an enormous volume. Therefore atoms are relatively far apart. This means that collisions between atoms are less frequent, which produces narrow hydrogen lines.

Now we draw an H-R diagram between luminosity and spectral class. Knowing both the spectral type and luminosity of a star would help an astronomer to instantly know where on the main sequence it lies. The H-R diagram is depicted below. It divides H-R diagram according to luminosity classes so that distinctions can be made between, for example, giant and super giant stars.

H-R diagram and stellar mass

H-R diagram is a graph between luminosity and temperature. Now we will see how to depict stellar masses on the H-R diagram. There are five methods mostly used for the determination of stellar masses. Mass and luminosity have been measured independently for many stars extending over a broad range of masses. It has been found that luminosity of a star is directly proportional to cube of its mass. i.e., $L \propto M^3$

It shows that luminosity strongly depends on mass. The masses of stars in the main sequence are shown in figure below. The above relation says that a star of mass $10M_{\odot}$ will have luminosity 1000 times greater than that of the Sun. Further the knowing L and M, we can predict how long a star will be in the main sequence. A star's lifetime is proportional to the mass available to burn and inversely proportional to the rate at which mass is used up.

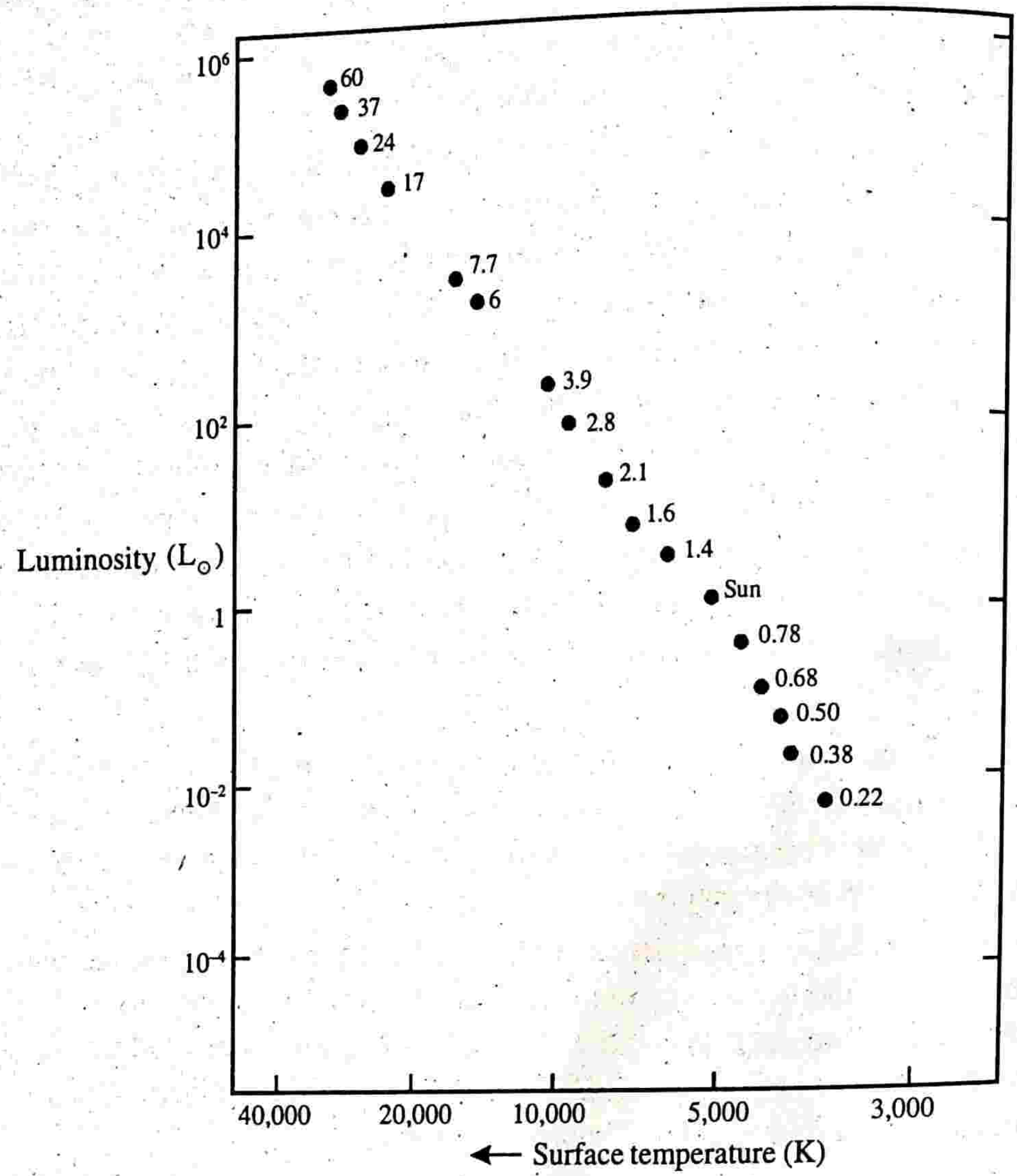


Figure 3.6: Mass and the main sequence (star's mass in terms of $M_\odot$)

i.e. $$\text{Life time} \propto \frac{M}{L} = \frac{M}{M^3} = \frac{1}{M^2}$$

$$\text{Life time} \propto \frac{1}{M^2}$$

That is life time of a star is inversely proportional to the square of its mass.

The more massive a star, the shorter its life time. Sun has a life time of 10 billion years. So a star of 10 solar mass will have a life time of 100 million years whereas the least massive stars $\left(=\frac{1}{10}M_\odot\right)$ last for trillion years.

IMPORTANT FORMULAE

1. The relation between distance and parallax

$$d = \frac{1}{p}$$

Where d is in parsecs and p is in arc seconds.

2. Relation between AU, ly and pc

(i) $1\,AU = 1.496 \times 10^{11}\,m$

(ii) $1\,ly = 9.46 \times 10^{15}\,m$

(iii) $1\,pc = 3.08 \times 10^{16}\,m$

$1\,ly = 6.32 \times 10^{4}\,AU$

$1\,pc = 2.05 \times 10^{5}\,AU$

$1\,pc = 3.26\,ly$

3. Relation between luminosity (L), apparent brightness (b) and distance d

$$L = 4\pi d^2 b$$

4. Relation between apparent brightness and magnitudes

$$m_2 - m_1 = -2.5 \log \frac{b_2}{b_1}$$

or

$$\frac{b_2}{b_1} = 10^{0.4(m_2 - m_1)}$$

5. Relation between apparent magnitude (m) absolute magnitude (M) and distance (d)

$$m - M = 5(\log d - 1)$$

6. Wien's displacement law

$$\lambda_m T = 2900000\,nmK$$

7. Flux (F), luminosity (L) and radius (R)

(i) $F = \sigma T^4$, $\sigma = 5.67 \times 10^{-8}\,Wm^{-2}K^{-4}$

(ii) $L = FA = 4\pi R^2 \sigma T^4$

8. Luminosity (L) of star is proportional to M^3

i.e. $L \propto M^3$

9. Lifetime of a star $\propto \frac{1}{M^2}$

UNIVERSITY MODEL QUESTIONS

Section A

(*Answer questions in about* **two** *or* **three** *sentences*)

Short answer type questions

1. What is astronomy?
2. What is astrophysics?
3. Distinguish between astronomy and astrophysics.
4. What is parallax?
5. What is stellar parallax?
6. Write down the relation between parallax and distance and explain the symbols used.
7. What is the limitation of parallax method?
8. Define luminosity of a star. What is its unit?
9. Define apparent brightness of a star.
10. Distinguish between luminosity and apparent brightness.
11. What are the factors on which apparent brightness depend?
12. What do you understand by magnitude of stars?
13. Write down the relation between change in magnitudes and apparent brightness of stars and explain the symbols used.
14. Distinguish between -apparent magnitude and absolute magnitude.
15. Write down the relation between apparent magnitude, absolute magnitude and distance.
16. What is the relation between colour and temperature of stars?
17. Distinguish between luminosity and flux.
18. Write down the relation between luminosity, flux and size of a star and explain the symbols used?
19. What is a star? What are its constituents?
20. What are the different spectral classes of stars?
21. Write down the spectral classes in accordance with increasing temperature?
22. Our sun is a G2 star. Explain.
23. Why astronomers study about stellar masses?
24. What are main sequence star?
25. What are the sources of energy of main sequence stars?
26. Write down three characteristics of main sequence stars?
27. What is an H-R diagram?
28. What is the importance of H-R diagram?
29. What is the relation between luminosity and mass of a star?
30. Draw an H-R diagram and indicate stellar radius.

Section B

(*Answer questions in a* **paragraph** *of about half a page to one page*)

Paragraph / Problem type questions

1. Explain stellar parallax method.
2. Discuss one method of distance measurement other than stellar parallax.
3. Obtain the relation between apparent magnitude and apparent brightness.
4. Obtain the relation between apparent magnitude, absolute magnitude and distance.
5. Obtain the relation between luminosity, flux and size of stars.
6. In an H-R diagram what are the informations that you can arrive at regarding luminosity.
7. If the parallax of the star Shravana is 0.198 arc second, calculate its distance from the earth in light year. [16.473 *ly*]
8. The distance of Bernard's star from the earth is 5.94 *ly*. Its luminosity is 1.404×10^{24} W. Find it apparent brightness. [3.53×10^{-11} Wm^{-2}]
9. The luminosity of the star Regulus is 5.46×10^{28} W, its brightness is 7.17×10^{-19} Wm^{-2}. Calculate the distance of star from the earth in parsec [25.32 pc]
10. The apparent magnitude of the star sirius A is –1.44 and that of Regulus star is +1.36 on the magnitude scale of stars. Calculate the relative brightness of the star Sirius A with respect to Regulus [13.18]
11. The dimmest star visible to the naked eye has a magnitude of 6. Compare its brightness with that of planet Venus whose magnitude is –4 [Venus is 10^4 times brighter]
12. The phenomenon of Nova involves the sudden outburst of a star. The star then becomes much brighter than usual for a few days. In 1995, a Nova appeared in the constellation of cygnus. In two days, the magnitude of the star changed from +15 to +2. By what factor did it brightness increase [158493]
13. The luminosity of the star Betelgeuse in the Orion constellation is 10,000 times that of the sun and its surface temperature 3000 K. Calculate the radius of the Betelgeuse with reference to that of the Sun. Take $T_\odot = 5800K$ [$R = 373.6 R_\odot$]
14. The apparent magnitude of Aldebaren is 0.80. If it is at 21 parsecs away from the earth. Calculate its absolute magnitude. [–0.811]
15. A star whose apparent magnitude 10 is located at a distance of 32.6 *ly*. What is its absolute magnitude? [M = 10]
16. If the apparent and absolute magnitudes of Sirius B are 8.6 and 11.4 respectively. Calculate its distance in AU. [5.646×10^5 AU]
17. Find out the ranges of temperature corresponding to which a star will appear red and yellow respectively. 620 to 770 nm - red and 500 to 600 nm for yellow.

$$\begin{bmatrix} \text{For red } 3766.6 - 4677\text{ K} \\ \text{For yellow } 4833 - 5800\text{K} \end{bmatrix}$$

18. If the luminosity of the white dwarf is 0.01 $L_\odot$ and its radius is 650 km. Calculate its temperature. $R_\odot = 6.96\times10^8$ m and $T_\odot = 5800K$ [160030 K]

19. The luminosity of the Sun is 3.9×10^{26} W and the value of solar constant on the surface of earth is $1388 Wm^{-2}$. Calculate the distance of earth from the Sun. [1.495×10^{11} m]

20. Suppose the Sun is taken from its present position which is at a distance of 1.496×10^{11} m from the earth to a new position located at 4 light years from earth. Calculate ratio of the apparent brightness of the Sun in the new position to the actual position.

[1.56×10^{-11}]

Section C

(Answer questions in about **one** to **two** *pages)*

Long answer type question (Essay)

1. Sketch an H-R diagram and write down all informations that we obtain from it.

Hints to problems

1 to 6 See book work

7. $d = \frac{1}{p}$, $p = 0.198$ d will be in parsec. $1pc = 3.26ly$

8. $b = \frac{L}{4\pi d^2}$ $L = 1.404\times10^{24}$, $d = 5.94\times9.46\times10^{15}$

9. $L = 4\pi d^2 b$ $\therefore$ $d = \left(\frac{L}{4\pi b}\right)^{\frac{1}{2}}$

10. We have $m_2 - m_1 = 2.5\log\frac{b_1}{b_2}$, $m_2 = 1.36$, $m_1 = 1.44$ get the result

11. Use $m_2 - m_1 = 2.5\log\frac{b_1}{b_2}$ find $\frac{b_1}{b_2}$

12. Use $m_2 - m_1 = -2.5\log\frac{b_2}{b_1}$

$m_1 = 15$, $m_2 = 2$ find $\frac{b_2}{b_1}$

13. $$L = 4\pi\sigma R^2 T^4$$

$$L_\odot = 4\pi\sigma R_\odot^2 T_\odot^4$$

$$\therefore \quad \frac{L}{L_\odot} = \frac{R^2}{R_\odot^2}\frac{T^4}{T_\odot^4}$$

$$\therefore \quad \frac{R}{R_\odot} = \left(\frac{L}{L_\odot}\right)^{\frac{1}{2}}\left(\frac{T_\odot}{T}\right)^2, \quad \frac{L}{L_\odot} = 10^4$$

$$\frac{T_\odot}{T} = \frac{5800}{3000} = 1.933, \quad \text{find } \frac{R}{R_\odot}$$

14. $m - M = 5(\log d - 1)$

$m = 0.80$, $d = 21$ parsec, find M

15. $m - M = 5(\log d - 1)$ $m = 10$, $d = \dfrac{32.6}{3.26}$ pc Find M.

16. $m = 8.6$, $M = 11.4$

Using $m - M = 5(\log d - 1)$, $d = 2.754$pc. $d = 2.754 \times 2.05 \times 10^5$ AU

17. Use $\lambda_m T = 2900000$nm

$$T = \frac{2900000}{\lambda_m}\text{nm}$$

For red $$T = \frac{2900000}{620} = 4677.4\text{K}$$

$$T = \frac{2900000}{770} = 3766.6\text{K}$$

For yellow $$T = \frac{2900000}{500} = 5800\text{K}$$

$$T = \frac{2900000}{600} = 4833\text{K}$$

18. $L_w = 4\pi\sigma R^2 T_w^4$

$$L_\odot = 4\pi\sigma R_\odot^2 T_\odot^4$$

$$\frac{L_w}{L_\odot} = \left(\frac{R_w}{R_\odot}\right)^2 \left(\frac{T_w}{T_\odot}\right)^4, \quad \frac{L_w}{L_\odot} = 0.001 = \frac{1}{100}$$

$$\frac{R_w}{R_\odot} = \frac{650\times10^3}{6.96\times10^8} = 93.39\times10^{-5}$$

Calculate T_w

19. Use $L = F\times 4\pi d^2$, $\quad F = 1388\,\text{Wm}^{-2}$ (given)

$$d = \left(\frac{L}{F4\pi}\right)^{\frac{1}{2}} = \left(\frac{3.9\times10^{26}}{1388\times4\times3.14}\right)^{\frac{1}{2}}$$

20. $L = 4\pi d^2 b$

$L_\odot = 4\pi d_\odot^2 b_\odot \quad L_\odot = 4\pi d_\odot'^2 b_\odot'$, equating we get

$$\frac{b'_\odot}{b_\odot} = \frac{d_\odot^2}{d_\odot'^2} = \left(\frac{1.496\times10^{11}}{4\times9.46\times10^{15}}\right)^2 = 1.56\times10^{-11}$$

4 STELLAR EVOLUTION

Introduction

Like other animate objects in this universe, a star is born, lives for a certain time and finally dies. **Stellar evolution is the process by which a star undergoes a sequence of radical changes during its life time.** During stellar evolution following changes are expected to follow protostar, premain sequence stage, main sequence stage. During the final stage star may undergo a nova or supernova explosion to become a red giant or supergiant. Depending upon the mass of the star it may end up with white dwarf or neutron star or a blackhole.

Birth of a star

When we look at a clear night sky we can see the gas clusters here and there. It contains mostly hydrogen (about 75%) 24% helium and the remainder such as N_2, O_2, CO_2, etc. called metals. If by accidently some denser part of the gas cluster occurs, then the gas particles, surrounding it are being attracted towards the denser part due to gravitational force. Through millions of years millions and millions of hydrogen atoms accumulate over a region. Hence forms a sphere of gas. This tight cluster of atoms formed by accident and held in grip of its own gravity called a protostar.

As the proto star consists of millions of hydrogen atoms it experiences a strong gravitational force. Because of this gravitational force protostar begins to contract. **It is called embryo star.** This contraction will go on till it becomes luminous. As the embryo star contracts, atoms of the gas collide with one another frequently with greater and greater speeds. The gas heats up. Eventually the gas will be so hot that when the hydrogen atoms collide they no longer bounce of each other but coalesce to form helium or we can say that hydrogen atoms undergo atomic reactions. In other words, a protostar stops collapsing when the core temperature becomes high enough to trigger hydrogen fusion. As the material pull it together into a ball, the pressure of the gas increases. The gas at the centre of the ball becomes extremely hot when the temperature at the centre reaches about 10 million kelvin. At this temperature hydrogen fusion can occur efficiently by the proton-proton chain reaction. The moment the ignition fusion process occurs will halt any further gravitational collapse of the protostar. **The star's interior structure stabilizes with the thermal**

energy created by nuclear fusion maintaining a balance between gravity and radiation pressure. This important act of balancing is called gravitational equilibrium. It is also sometimes referred to as hydrostatic equilibrium. The star is now a hydrogen burning main sequence star.

The time taken for the formation of a protostar to the birth of a main sequence star depends on the mass of the star. When the mass of the star increases its life time decreases. For example a high mass protostar may collapse in million years or less while a star with $1M_\odot$ life for 50 million years. This is because $L \propto M^{3.5}$.

The changes that occur to a protostar's luminosity and surface temperature can be shown on a special H-R diagram. This is known as an evolutionary track or a life track of a star. Each point along the stars track (see figure below) represents its luminosity and temperature at some point during its life and so it shows how the protostars appearance changes due to changes in its interior. In figure 4.1 evolutionary tracks for seven protostars of different masses from $0.5M_\odot$ to $15M_\odot$ are shown.

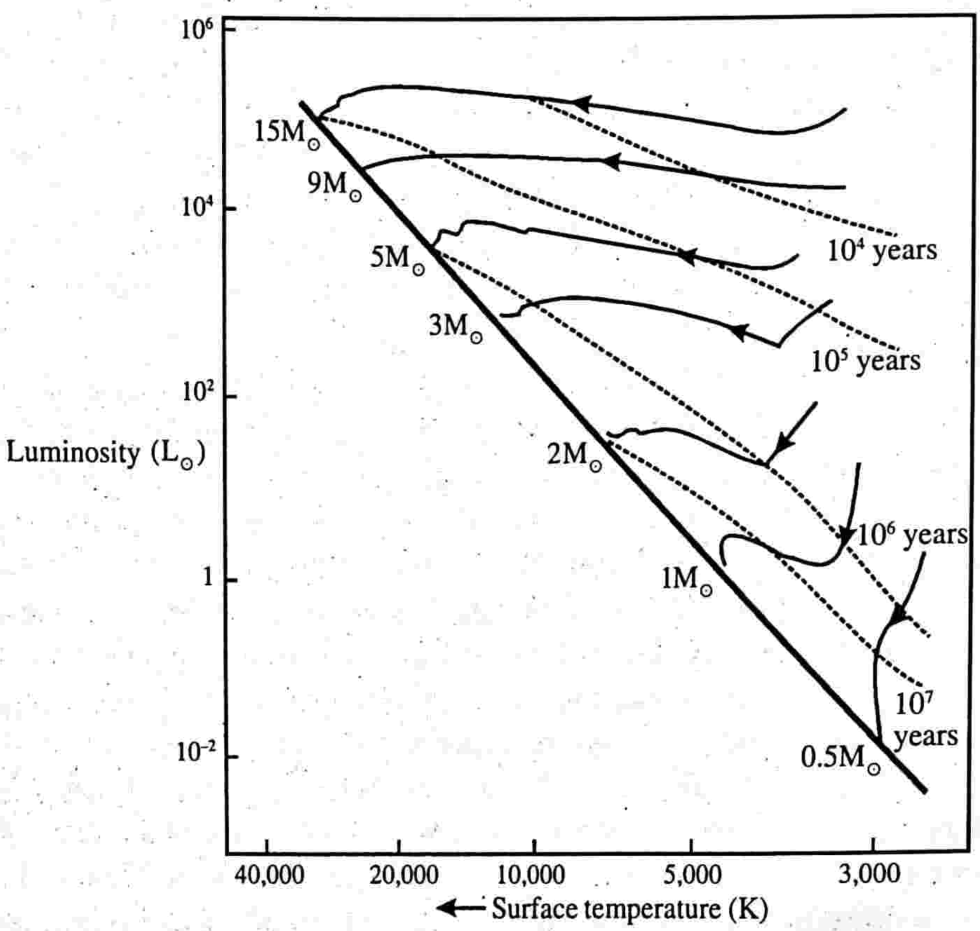


Figure 4.1: Premain-sequence evolutionary tracks (Solid line with arrow mark are evolutionary tracks)

These evolutionary tracks are theoretical models developed by the Japanese astrophysicist C. Hayashi and the phase of a protostar under goes before it reaches the main sequence is called Hayashi phase. Since protostars are relatively cool so the tracks begin at the right side of the H-R diagram. However, the subsequent evolution is very different for stars of differing mass. The masses shown in the H-R diagram is the final mass when it becomes a main sequence star. Also note that the greater the mass, the higher the temperature and luminosity.

A protostar undergoes 4 distinct phase changes before it reaches the main sequence.

Different phases of a star in their life track

Phase 1

The protostar forms from a cloud of cold gas, thus the evolutionary track starts from on right most side of the H-R diagram. However, its surface area is enormous so its luminosity can be very large $(L \propto 4\pi R^2)$. This luminosity may be 100 times more than luminous when it becomes.a star.

Phase 2

It is due to its large luminosity, the young protostar rapidly loses it energy generated via gravitational collapse. So further collapse proceeds at a relatively rapid rate. Its surface temperature increases slightly during the next million years, but its diminishing size reduces the luminosity. The evolutionary track now progresses almost vertically downward on the H-R diagram.

Phase 3

Now the core temperature has reached 10 million kelvin, hydrogen nuclei fuse into helium. However the rate of nuclear fusion is not sufficient to stop the gravitational collapse of the star. As a result star shrinks and its surface temperature increases. The process of shrinking and heating will result in a small increase in luminosity over the next 10 million years. The evolutionary track now progresses leftward and slight upward on the H-R diagram.

Phase 4

Both the rate of nuclear fusion and the core temperature increase over the next tens of millions of years. Once the rate of fusion is high enough to balance the gravitational collapse, gravitational equilibrium is achieved. The result is that the star settles onto the hydrogen burning main sequence star - a youngster star. (See figure 4.2)

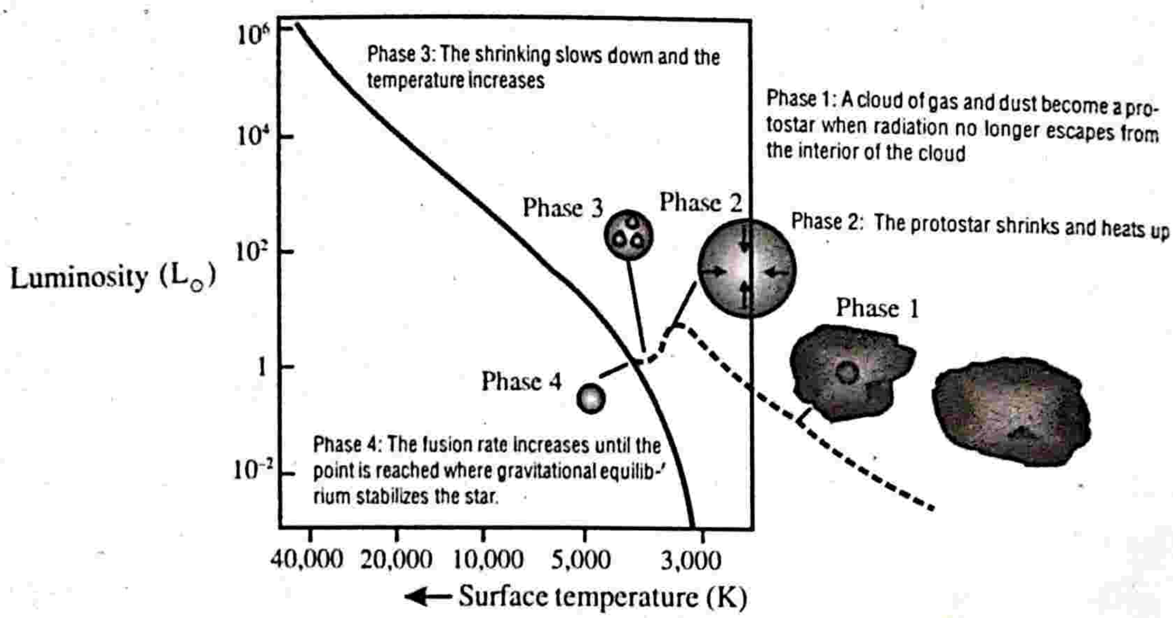


Figure 4.2: The evolutionary track of a protostar with four phases

Note: Four phases of the evolutionary track is discussed by taking $1M_\odot$.

Premain-sequence evolution and the effect of mass

In the last section we found that how a cloud can contract and become a protostar. Protostars have different masses. Here we discuss how masses affect the evolution of protostar before reaching the main sequence.

We begin with a protostar leads to a mass $1M_\odot$ (a star just like Sun). The outer layers of such a protostar are cool and opaque. Since it is opaque the energy released as radiation due to the shrinkage of the inner layers cannot reach the surface. Thus energy moves to the outer layer by convection process. This process is less efficient and slower. As a result of this the temperature remains more or less constant but luminosity decreases as radius decreases due to shrinkage. Thus the evolutionary track moves downward on the H-R diagram. See evolutionary track corresponding to $1M_\odot$ in figure 4.2.

The surface temperature is more or less constant but the conditions inside is changing. The internal temperature increase which reduces the opacity within the protostar and allows the transfer of energy by radiation in the interior regions and by convection in the outer layers. This process is ongoing within the sun now. The net result is that the energy can escape much more easily from the protostar and the luminosity increases. Thus the evolutionary track bending upward and moves left. After an interval of a few million years the temperature within the protostar is high enough - 10 million kelvin - for nuclear reaction to begin. Eventually heat and associated internal pressure increases. This radiation pressure acting outward can balance the gravitational contraction of the star, there by star reaching hydrostatic equilibrium. Now the protostar has reached the main sequence - It is now a main sequence star.

A protostar with a mass of about or greater than $4M_\odot$ contract and heat up at a more rapid rate and so the hydrogen burning phase begins earlier. The net result is that the luminosity stabilizes at approximately its final value, but the surface temperature continues to increase as the protostar continues to shrink. In the H-R you can see nearly constant luminosity (almost a straight line). This is also true for $9M_\odot$ and $15M_\odot$.

An increase in mass will result in a corresponding increase in pressure and temperature in the interior of a star. In very massive stars there is a much greater temperature difference between the core and the outer layers. This allows convection to occur much deeper into the stars interior regions. In contrast, the massive star will have very low density out layers, so energy flow in these regions are easily performed by radiative process.

The main difference between stars of mass greater than $4M_\odot$ and less than $4M_\odot$ is that a star with a mass greater than $4M_\odot$ will have convective outer layers.

Stars with mass less than $1M_\odot$ have a very different internal structure. In these protostars, the interior temperature of the protostar is insufficient to ionise the inner region which is thus too opaque to allow energy flow by radiation. The only way to transport energy is the convection process. The radiative, convective and mass dependence are shown in figure 4.3.

If may be noted that all the evolutionary tracks end at the main sequence where nuclear reactions produce energy by converting hydrogen into he-

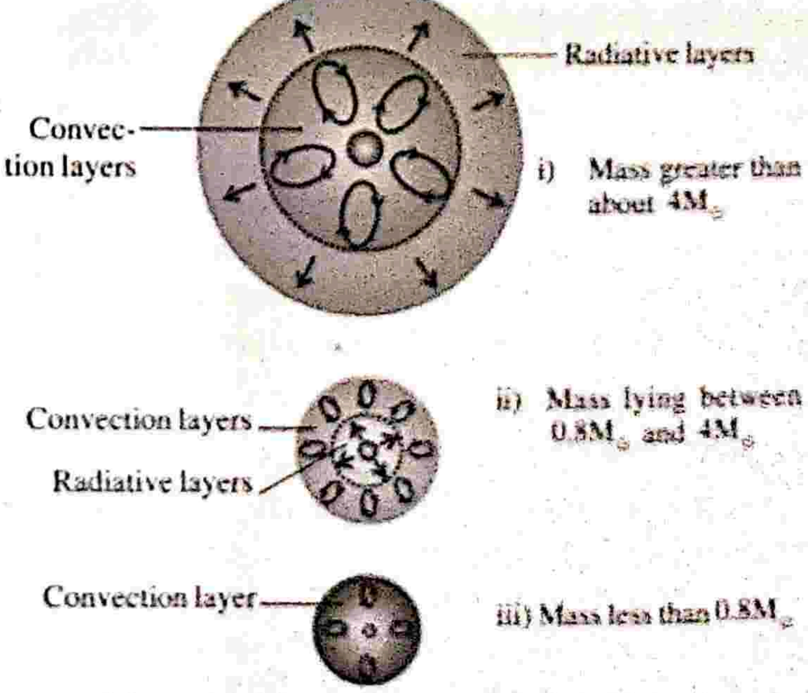


Figure 4.3: Mass and energy flow dependence

lium. Moreover almost all stars in the main sequence are in hydrostatic equilibrium. Thus main sequence is represented by mass-luminosity relationship. The greater the mass the greater the luminosity. The luminosity versus mass is plotted in figure 4.4. A star of mass $10M_\odot$ has about $300L_\odot$.

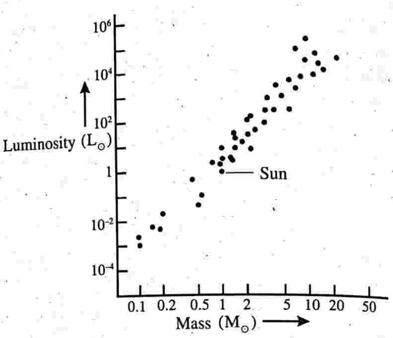


Figure 4.4: Mass-luminosity graph

For the stars on the main sequence, there is a ratio between the mass and the luminosity. This ratio gives us the life time of the star in the main sequence.

One important point to be noted is that when a star reaches on the main sequence almost all stars spend most of lives on the main sequence. That is stars live 90% of their lifetimes as a younger star. The remaining 10% time is spent in the aged stage such as giant, supergiant white dwarf, neutron stars and blackholes

Finally we say that masses of stars have limits. It has been theoretically deduced that stars above 150-200$M_\odot$ cannot form. If a star of mass above $150M_\odot$ is formed it generates so much gravitational energy which cannot be balanced by the radiation pressure. Such stars undergo implosion and tear themselves apart. There is also a lower limit for the formation of stars. It is less than $0.08M_\odot$. The stars with a mass less than $0.08M_\odot$ can never achieve the 10 million kelvin core temperature necessary to initiate nuclear fusion. Such stars are called failed stars. They slowly radiate their energy till it becomes cool. These objects are called brown dwarfs and seem to occupy between planet and stars. Brown dwarfs radiate energy in the infrared region, so it is very difficult to detect them. A brown dwarf of mass $0.05M_\odot$ was detected in the year 1995 and named as Gliese 229 B.

Galactic clusters

Usually stars do not form in isolation. There is an exemption forth this, in 2006 it has been concluded that stars can also form in isolation. Here we discuss about star clusters. We found that stars are formed from nebulae which is a sphere of gas containing hydrogen, nitrogen, metals, dust particles etc. A nebulae can contain materials that could form hundreds of stars. So stars tend to form in groups or clusters. The stars form out of the same nebulae need not have the same mass. That is stars may

have different masses. It is due to the difference in mass they reach the main sequence at different times. This is because high mass stars evolve faster than low mass stars. For example, a $1M_\odot$ protostar takes about 20 million years to become a main sequence star while $12M_\odot$ star may take only 20,000 years when high mass stars are shining brightly as stars, the low mass stars may not have begun shining. In this situation the intense radiation emitted by bright stars may disturb the normal evolution of the lower mass stars and so reduce their final mass.

When time passes the group of stars gradually get dispersed. Since the massive stars have much shorter life times they may not be able to escape from their birth place, where as smaller stars which have long life time may escape from their birth place.

If a cluster containing several thousand stars, their combined gravitational pull towards the centre will slow down the dispersion of the group. Thus clusters can be divided into two, globular clusters and open clusters. Open clusters are called galactic cluster. The closed clusters contain the oldest population of stars where as galactic clusters contain youngest star population.

A small collection of gravitationally bound younger stars containing 10 to 100 stars without having particular shape is called a galactic cluster. Stars in the galactic clusters can be distinguished. Pleiads (Karthika) and Hyades (Rohini) are examples of galactic clusters. The members of galactic cluster need not have the same brightness. Some are bright and some are faint stars. **The stars that makeup a galactic cluster are called population I stars,** which are metal rich and usually found in or near the spiral arms of the galaxy. The size of galactic cluster can from a few dozen *ly* to about 70 *ly*.

Galactic clusters are dissimilar in appearance and vary in character. This is due to their circumstances of birth. It is the interstellar cloud that determines both the number and types of stars that are born. Factors such as size, density, turbulence, temperature and magnetic field all play a role as the deciding parameters in star birth. In the case of giant molecular clouds (GMC) the conditions can give rise to both O and B type stars along with solar type dwarf stars, where as in small molecular clouds (SMC) only solar type stars will be formed. An example of SMC is the taurus dark cloud which lies beyond the pleiads star cluster.

By observing a star cluster we can study the process of star formation and the interaction between low and high mass stars. For example see H-R diagram for NGC 2264 cluster located in monoceros (unicorn) constellation. From the diagram it can be seen that the hottest star with a temperature of about 20,000 K almost reached main sequence while stars with temperatures about 10000 K and below have not reached main sequence. It is concluded that this cluster is very young only two millions years old and it is at a distance of 800 pc from earth.

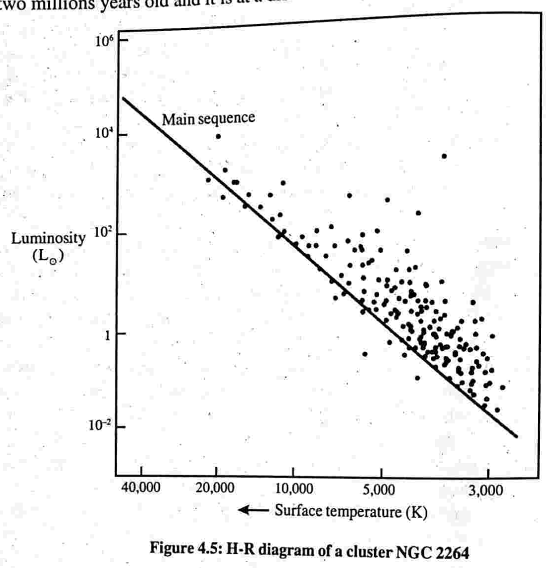


Figure 4.5: H-R diagram of a cluster NGC 2264

So far thousands of galactic clusters have been discovered, only a few are observed to be at distances greater than 25° above or below the galactic equator. Some parts of the sky are very rich in clusters. The constellations cassiopea and puppis (Amaram) contain large number of clusters.

Star formation triggers

Here we discuss about what are the causes of protostar formation. There are three mechanisms by which star formation triggers. They have dissimilar origins.

They are

(i) The spiral arms of a galaxy

(ii) Expanding H II regions

(iii) Supernovae

The spiral arms of a galaxy

We found that spiral arms of galaxies are a prime location for star formation. This is because the gas and dust clouds temporarily pile up as spiral arms orbit around the centre of galaxy. In such a spiral arm the molecular clouds are compressed as it passes through the region. In the molecular cloud's densest regions vigorous star formation occurs.

Expanding H II regions

Massive stars (O type and B type) emit immense amount of ultraviolet radiation. This causes the surrounding gas to ionise and an H II region is formed within the molecular cloud. The strong stellar wind and ultraviolet radiation curve out a cavity within the molecular cloud into which the H II region expands. The stellar wind is moving at supersonic speed as a result a shock wave associated with this supersonically expanding H II region then collides with the nest of the molecular cloud. This process compresses the cloud there by star formation begins to occur. This mechanism is occurring in orion nebula. The four stars of the trapezium are ionising the surrounding material. The nebula is located at the edge of a giant molecular cloud of mass 5,00,000 $M_\odot$.

Supernova

Supernova is a catastrophic explosion of a star in its old age. In this explosion the out layers of the star are tear into pieces and thrown away. The ejected outer layers move in space with an incredible speed, may be several thousand kilometres per second results in shock waves. This shock waves impact the material in the interstellar medium triggering further star formation.

The Sun

The Sun is the nearest star to us which lies on the main sequence. The Sun has been in the main sequence for the last 50 million years and it will remain there for another 50 million years. Here we discuss on the internal structure, the means of energy production and the manner in which energy transported from its source to us on earth.

The internal structure of the Sun

The internal structure of the sun mainly comprises 5 parts. They are (i) photosphere (ii) convection zone (iii) plasma zone (iv) radiation zone and (v) core

(i) Photosphere

It is the visible surface of the sun which has a temperature of 5000 K. Though it looks like a well defined surface from the earth, it is a gas less dense than the earth's atmosphere. Both the density and temperature steadily increase as we go towards the core of the Sun.

(ii) Convection zone

Convection zone is the very turbulent area beneath the photo sphere. In this area energy generated in the core of the Sun travel upward, transported by rising the columns of hot gas and the falling of cool gas taking place. This process is called convection, hence the name convection zone.

(iii) Plasma zone

Descending deeper through the convection zone the pressure and density increase quite substantially along with temperature. When we reach at a stage where the gas is under extreme conditions of pressure and temperature becomes ionised. This region is called plasma region. Plasma is a collection of positively charged ions and free electrons. The temperature of this region is about 2 million kelvin. This region absorbs photons. The density of this region is far greater than that of water.

(iv) Radiation zone

This region is in the more stable plasma state and one third of the way down to the centre. The temperature of this region is about 10 million kelvin. Here the energy is transported outward primarily by photons of X-ray radiation.

(v) Core

The central region of the Sun is called the core of the Sun. The temperature is about 15 million kelvin. This is the region where nuclear process takes place. Hence it is also called as nuclear burning zone. Here hydrogen is being converted into helium. The density of this region is about 150×10^3 kg m^{-3}. This means that the density is about 150 times that of water. The radius of the inner core is about 25% of the total radius of the sun. Beyond this radius (upward) practically no nuclear reaction takes place.

See also the figure given below.

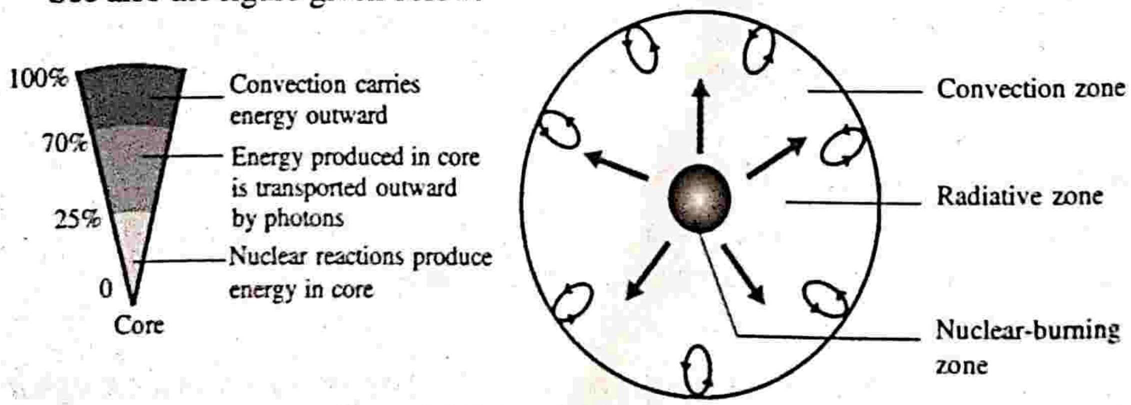


Figure 4.6: Internal structure of the sun

The inner core of the sun contains only about 34% hydrogen whereas the outer core contains 71% hydrogen. The reason is that for the last 4.6 billion years hydrogen has been converting into helium. The luminosity of the Sun is 3.8×10^{26} joules per second. If we could somehow capture all of the energy for one second, it would be sufficient to meet all current energy demands for the human community for the next 50,000 years. But remember that only a tiny fraction of this reaches the earth.

According to the current model of energy production in the Sun, nuclear fusion is the source. With this energy the Sun will shine for 10 billion years. Among this 4.6 million years elapsed and the remaining 5.4 million years further to go by the Sun. According to this model the size of the Sun is stable. This stability is maintained by balancing the gravity pull acting inward and radiation pressure acting outward.

The proton-proton chain reaction

The main source of the energy of the sun is through the process of nuclear reaction. Sun contains immense amount of the hydrogen. The fusion of hydrogen into helium releasing energy was first proposed by the British astronomer Arthur Eddington in 1920 but the details of the reaction were came out only after 1940.

Hydrogen nucleus contains only one proton whereas helium nucleus contains two protons and two neutrons. So four protons are required to produce a helium nucleus. Collisions of four protons never make a helium nucleus, because no such reaction is found to occur in laboratories. The actual reason is that when protons collide it is due to mutual repulsion they move apart and cannot involve in reaction. For any reaction to occur there are some conditions required. One is that protons have to move with very high velocity. This condition exists in the centre of the Sun. At the core of the Sun, the temperature is about 15 million kelvin and the proton will be travelling at about 1 million kilometers per hour. Even at this fantastic speed the probability of occurring reaction is very small. A single proton would take nearly about 5 billion years to get reacted with another photon. However, as there are so many protons in the Sun's core, every second 10^{34} of them can undergo a reaction.

Now we believe that proton and proton undergo reaction in a series of manner involving two reactions at a time. This is called proton-proton chain reaction. This reaction involves three steps.

Step 1

Two protons fuse to form a deutirium nucleus, a positron and a neutrino.

i.e. $$^{1}H + {}^{1}H \rightarrow {}^{2}H + \beta^{+} + \upsilon$$

The positron β^{+} (a positively charged electron) does not last long, it will soon meet up an electron producing two gamma rays that are absorbed by the surrounding gas. Neutrino rarely interacts with matter and so pass straight out into space.

Step 2

The deuterium now fuses with a proton producing a helium nucleus (^{3}He) and gamma rays. The ^{3}He nucleus consists of consists of two protons and one neutron.

i.e. $$^2H + {}^1H \rightarrow {}^3He + \gamma$$

Step 3

Finally two ^{3}He nuclei combine to form ^{4}He nucleus and give back two protons.

i.e. $$^3He + {}^3He \rightarrow {}^4He + 2\,{}^1H$$

So the net result of reaction is

$$6\,{}^1H \rightarrow {}^4He + 2\,{}^1H$$

The total energy released in this process worked out is 26.72 MeV. So the complete reaction can be written as

$$6\,{}^1H \rightarrow {}^4He + 2\,{}^1H + 24.72 \text{ MeV}.$$

Calculation of energy released

In proton-proton chain reaction though 6 protons take part, it gives two proton, only 4 protons converted into ^{4}He nucleus.

Mass of one proton, $m_p = 1.6726 \times 10^{-27}$ kg

Mass of four protons $= 4 \times 1.6726 \times 10^{-27}$ kg

$= 6.6904 \times 10^{-27}$ kg

Mass of helium nucleus $= 6.645 \times 10^{-27}$ kg

$\therefore$ The difference in mass $= 6.6904 \times 10^{-27} - 6.645 \times 10^{-27}$

$$\Delta m = 0.0454 \times 10^{-27} \text{ kg}.$$

This difference in mass is converted into energy according to Einstein's mass-energy relation $E = mc^2$.

$\therefore$ The energy released, $\Delta E = \Delta mc^2 = 0.0454 \times 10^{-27} \times (3 \times 10^8)^2$

$$\Delta E = 0.4086 \times 10^{-11} \text{ J}$$

Converting this into MeV, we get

$$\Delta E = \frac{0.4086 \times 10^{-11}}{1.6 \times 10^{-13}} \text{MeV}$$

$$\Delta E = 25.54 \text{ MeV}.$$

It may be noted that the Sun converts about 600×10^9 kg of matter into 596×10^9 kg of helium in every second. The missing 4×10^9 kg of matter are converted into energy. A rough calculation shows this figures.

We found that 4 protons having mass 6.6904×10^{-27} kg is converted into helium of mass 6.645×10^{-27} kg.

$$\therefore \qquad \frac{\Delta m}{m} = \frac{0.0454 \times 10^{-26}}{6.6904 \times 10^{-27}} \simeq 0.7\%.$$

Let M be amount of hydrogen converted into helium in every second. Out of which only 0.7% is converted into energy. i.e. $M \times \frac{0.7}{100} c^2$ energy is released. But the energy released by the Sun in one second is 3.8×10^{26} J s^{-1} (Luminosity of Sun)

$$\therefore \qquad M \times \frac{0.7}{100} \times (3 \times 10^8)^2 = 3.8 \times 10^{26}$$

or

$$M = \frac{3.8 \times 10^{26} \times 100}{0.7 \times (3 \times 10^8)^2} \simeq 600 \times 10^9 \text{ kg}.$$

0.7% of this is $600 \times 10^9 \times \frac{0.7}{100} \simeq 4 \times 10^9$ kg is converted into energy.

$\therefore$ Mass converted into helium is $600 \times 10^9 - 4 \times 10^9 = 596 \times 10^9$ kg

Energy transport from the core to the surface

Energy produced in the central region of the Sun in the form of X-ray photons flows outward towards the surface. If the Sun were transparent these photons reach the surface after 2 seconds. The Sun is transparent means there are no interactions between photons and electrons. If there were no interactions between photons and electrons, time taken by the photon

$$t = \frac{x}{v} = \frac{\text{radius of the sun}}{\text{speed of light}}$$

$$t = \frac{6.9 \times 10^8}{3 \times 10^8} = 2.3 \text{ s.}$$

i.e. light takes only 2.3 seconds to exit the Sun if there were no interactions.

But the Sun's gases are not transparent. So during the passage of photons, every now and then they collide with nearby electrons. The electron can actually absorb the photon and takes its energy. This causes the electron to jump out of its orbit to new higher orbit. Since the electron can only exist at specific energy levels, this usually results in the electron giving off a photon and goes back to its stable orbit. So technically a photon born in the middle of the Sun never makes it to the surface. It is always absorbed and re-emitted as another photon. Photons are continuously absorbed and re-emitted throughout the interior of the Sun. The emitted photon will not necessarily emitted in an outward direction but rather a totally random direction. This results in an apparent dance like motion of the photons inside the star. This slow dance like motion of photons is called random walk. By such a random walk photon travels along a zig-zag path from the centre of the Sun to its outer surface. Finally, the photon escapes from the Sun's surface. During their random walk, the photons which are X-ray photons at the core become optical photons by the time they reach the surface.

The above discussion shows that there is a considerable time delay before energy produced at the core reaches the surface. On an average photon takes about 1.7×10^5 years to reach the surface. **This time taken by a photon to leave out of the Sun is called photon diffusion time.**

A simple calculation confirms this long delay in time.

Photon diffusion time $\tau = \dfrac{\text{Total photon energy}}{\text{Luminosity}}$

$$\tau = \frac{1.4 \times 10^{39}}{3.8 \times 10^{26}} = 3.68 \times 10^{12} \text{s}$$

$$\tau = \frac{3.68 \times 10^{12}}{3.15 \times 10^{7}} \text{ years}$$

or $$\tau = 1.17 \times 10^5 \text{ years.}$$

The above discussion brings us two important informations. Firstly, when we observe sunlight, we know nothing about what is going on at the core at the moment of observation. This is because the light we see now created in the core some thousands of years ago. Secondly suppose Sun ceases to produce energy, we can notice this only after thousands of years. This implies that the brightness of Sun is very in sensitive to changes in the energy production.

Binary stars

The stars that may appear to the naked eye to be just one star, but on observation with either binoculars or telescopes resolves themselves into two stars are called binary stars or double stars. Many stars appear as double stars due to their position in the same line of sight as seen from the earth. These are called optical doubles.

Binary stars are actually pairs of stars gravitationally bound and revolve about a common centre of mass with a common period.

Classification of binary stars

Binary stars are classified into four. They are (i) spectroscopic binaries (ii) eclipsing binaries (iii) astrometric binaries and (iv) visual binaries.

(i) Spectroscopic binaries are binaries which cannot be resolved even by largest telescopes. These binaries can be distinguished by analysing their spectra.

(ii) Eclipsing binaries are binaries in which one of the stars moves during its orbit in front of its companion. The binary star Algol belongs to Perseus constellation is an example for eclipsing binary.

(iii) In astrometric binaries one star is too faint to be seen and its presence is inferred from the perturbations in the motion of the other star. The binary star Sirius belongs to α Canis major (big dog) constellation is an example for this.

(iv) Visual binaries are binary stars in which two are far enough apart to be seen separately by an optical telescope. In this one of them is bright and the other is faint. The brighter star is called primary star and the faint one is called secondary or companion star. The two stars in this binary can be observed visually hence the name.

The study of binary stars are very important. This is because by observing the period of motion of binary we can determine their mass, which will help us to determine the evolutionary processes of stars.

Here we discuss only about visual binaries. To locate visual binaries two important terms are to be studied. They are separation and position angle (PA).

Separation

Separation is the angular distance between the two stars measured in seconds of arc. This angle is measured from the brighter star to the fainter.

Position angle

Position angle is the relative position of the secondary star with respect to the primary star. It is measured in degrees. Now our aim is to depict separation and position angle of a binary in a graph. For this imagine a coordinate system whose origin is the primary star. The north direction is assigned as 0°, due east 90°, due south 180°, due west 270° and back to 0° as shown in figure.

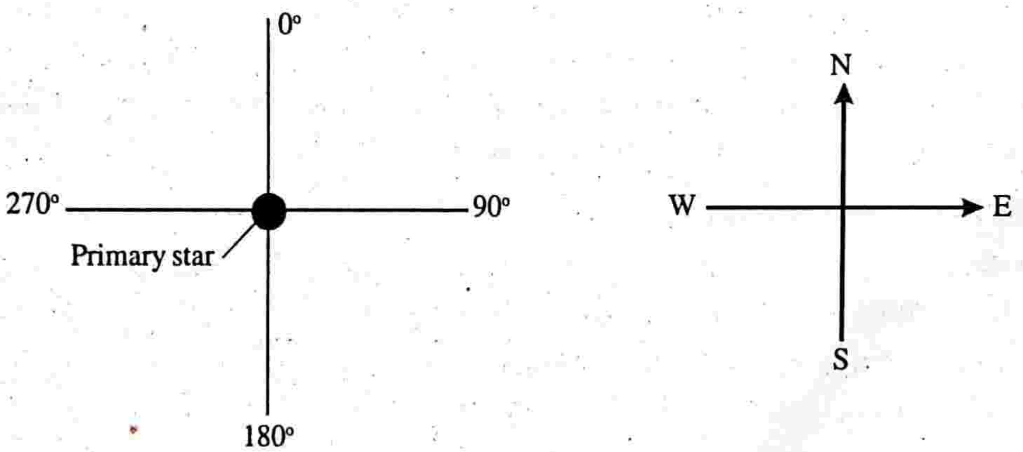


Figure 4.7

In each year calculate the separation and position angle and mark it on the graph. For example the separation and position angle of the star γ Virginis (a visual binary) is plotted below.

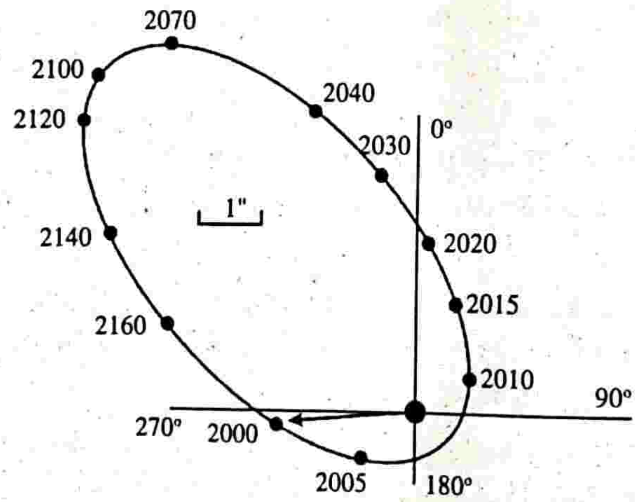


Figure 4.8: Separation and position angle of γ Virginis

From the graph it can be readily seen that in the year 2000 the separation between the two stars is 1.8″ at PA of 267°. It may also be noted that the separation and PA are constantly changing. Stars with long period of time will have no appreciable change in PA for several years.

Masses of orbiting stars

The masses of visual binary can determined by using Kepler's third law ($T^2 \propto a^3$) and Newton's law of gravitation. The first experimental evidence of Newton's law of gravitation $F = \frac{Gm_1m_2}{r^2}$ came from the observation of time period of binary stars.

Consider a binary system of two stars A and B of masses m_A and m_B (where $m_A > m_B$) forming a pair and bound by their mutual gravitational attraction. These stars are revolving around their common centre of mass (CM) of the system in their elliptical orbits such that CM remains constant. See figure below.

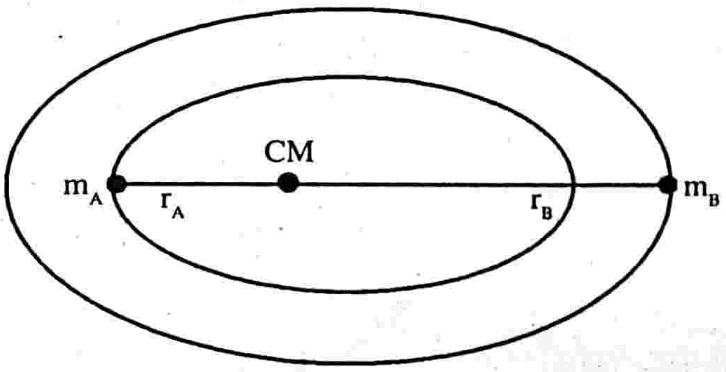


Figure 4.9

Let r_A be the distance of star A from the CM and r_B be the distance of star B from the CM. Then

$$r_A + r_B = r \qquad .....(1)$$

the separation between the two stars.

From the definition of centre of mass we have

$$m_A r_A = m_B r_B \qquad .....(2)$$

From eq (1), $\quad r_B = r - r_A$

Put this in equation (2), we get

$$m_A r_A = m_B(r - r_A)$$

or $$r_A = \frac{m_B r}{m_A + m_B} \qquad \text{..... 4(a)}$$

$$r = \frac{(m_A + m_B)}{m_B} r_A \qquad \text{..... 4(b)}$$

The gravitational force exerted on the star of mass m_A by the star of mass m_B is

$$F = \frac{G m_A m_B}{r^2}$$

If the path of the orbit is circular, the centripetal force required $(m_A r_A \omega^2)$ is supplied by the gravitational force $\left(F = \frac{G m_A m_B}{r^2}\right)$.

i.e. $$\frac{G m_A m_B}{r^2} = m_A r_A \omega^2 \qquad \text{..... (5)}$$

Substituting for r from eqn 2, we get

$$\frac{G m_A m_B}{\left(\frac{m_A + m_B}{m_B}\right)^2 r_A^2} = m_A r_A \omega^2$$

or $$\frac{m_B^3}{(m_A + m_B)^2} = \frac{r_A^3 \omega^2}{G}$$

Using $\omega = \frac{2\pi}{T}$ where T is the time period of the binary star.

$$\frac{m_B^3}{(m_A + m_B)^2} = \frac{r_A^3 4\pi^2}{T^2 G}$$

Substituting for r_A from eq 4(a), we get

$$\frac{m_B^3}{(m_A + m_B)^2} = \frac{m_B^3 r^3}{(m_A + m_B)^2} \frac{4\pi^2}{T^2 G}$$

or
$$T^2 = \frac{4\pi^2}{G(m_A + m_B)^2} r^3 \qquad \text{..... (6)}$$

This is the Kepler's third law $T^2 \propto r^3$.

or
$$m_A + m_B = \frac{4\pi^2}{GT^2} r^3 \qquad \text{..... (7)}$$

Knowing the time period (T) of the binary star and the separation between the two stars (r). The total mass of binary star can be determined. Determining one of the masses of binary, the other can be determined.

With much more rigorous calculation (see example 2), considering elliptical orbits, we get

$$m_A + m_B = \frac{4\pi^2}{GT^2} a^3 \qquad \text{..... (8)}$$

Where a is the semi major axis of the elliptical orbit of one star and T is the time period of one star to complete one rotation around the other.

If we express masses m_A and m_B in terms of solar mass $m_\odot$, a in astronomical unit and T in years, the value of $\frac{4\pi^2}{G}$ turns out to be 1. Thus we have

$$m_A + m_B = \frac{a^3}{T^2} \qquad \text{.....(9)}$$

Determination of T and a

To determine the total mass of the binary, we have to determine the time period of the one star with respect to the other and the semi major axis of the orbit. For this we determine the stars orbits by observing one star for several years. This may take a few years or tens of years, but we can eventually determine the time needed (T) for one star to completely orbit the other. Knowing the orbit of one star with respect to the other we can calculate the semi major axis.

Example 1

Consider the double star system Sirius A and Sirius B. The orbital period of the binary is 50 years and the semimajor axis is 19.8 AU. Determine their combined mass.

Solution

We have
$$m_A + m_B = \frac{a^3}{T^2}$$

$$m_A + m_B = \frac{19.8^3}{(50.1)^2} = \frac{7762.4}{2510}$$

$$\therefore \quad m_A + m_B = 3.09\ m_\odot$$

Example 2

A binary star system consists of two stars s_1 and s_2 with masses m and 2m respectively separated by a distance r. If both s_1 and s_2 individually follow circular orbits around the centre of mass with speeds v_1 and v_2. Calculate $\frac{v_1}{v_2}$.

Solution

From the figure we have

$$mr_1 = 2mr_2$$

$$r_1 = 2r_2$$

but
$$r_1 + r_2 = r$$

$$r_1 + \frac{r_1}{2} = r$$

$$r_1 = \frac{2}{3}r \quad \text{and} \quad r_2 = \frac{1}{3}r$$

For the star s_1

$$\frac{Gm2m}{r^2} = \frac{mv_1^2}{r_1}$$

$$\therefore \quad v_1^2 = \frac{2Gm}{r^2} r_1 \qquad \text{..... (1)}$$

For the star s_2

$$\frac{Gm2m}{r^2} = \frac{2mv_2^2}{r_2}$$

$$\therefore \quad v_2^2 = \frac{Gm}{r^2} r_2 \qquad \text{..... (2)}$$

$\frac{eq\,(1)}{eq\,(2)}$ gives

$$\frac{v_1^2}{v_2^2} = 2\frac{r_1}{r_2}$$

$$\frac{v_1}{v_2} = \sqrt{2\frac{r_1}{r_2}} = \sqrt{\frac{2\cdot\frac{2}{3}r}{\frac{r}{3}}} = 2$$

Lifetimes of main sequence stars

So far we have been discussing how a star is formed, how, the mass of stars can be determined and how long does it take to become a star etc. Now we shall discuss how long a star will remain on the main sequence and what are the changes in the internal structure of star.

All stars on the main sequence are fundamentally alike in their cores. This is because it is here that stars convert hydrogen into helium. **The process of converting hydrogen into helium is called core hydrogen burning.**

The time that stars spend on the main sequence, the stars with smaller masses live longer.

Main sequence lifetimes for stars of different masses are depicted in the figure given below.

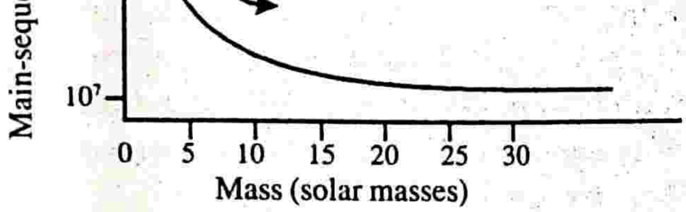


Figure 4.10: Main-sequence lifetimes for stars of different mass

It may also be noted that high mass stars are extremely bright this is because high mass stars use up their fuel hydrogen at a fast rate. This will in turn reduce their lifetime. For example O-type stars are much more massive than M-type stars. Thus O-type stars use their fuel at faster rate than M-type stars. Hence the luminosity of O-types stars higher and lifetimes are shorter compared to M-type stars. O-type stars have luminosity $80,000 L_\odot$ and lifetime 3×10^6 years whereas M-type stars have luminosity $0.08 L_\odot$ and lifetime $56,000\times10^6$ years. The mass, luminosity and life time of some stars are given in the table 4.1.

Table 4.1 Mass,, spectral class, and main-sequence lifetimes

Mass, $M_\odot$	Temperature, K	Spectral class	Liminosity, $L_\odot$	Main-sequence lifetime, 10^6 years
25	35,000	O	80,000	3
15	30,000	B	10,000	11
3	11,000	A	60	640
1.5	7000	F	5	3600
1	6000	G	1	10,000
0.75	5000	K	0.5	20,000
0.5	4000	M	0.08	56,000

Red giant stars

When stars are in the main sequence, they use up their hydrogen fuel and converting into helium producing huge amount of energy. This will go on occurring for millions and millions of years. When all the hydrogen has been used up, the energy production ceases. At this time star begins to use up its gravitational energy to supply its energy needs. Thus the core will start to cool down so the pressure decreases with the result that the outer layers of the star will begin to compress the inner core by its weight. The effect is to increase the temperature of the core and heat flows outward from the core. This heat formed is not due to nuclear reactions but due to conversion of gravitational energy into thermal energy.

In a short time the hydrogen layers surrounding the core become too hot to start nuclear reactions. **This conversion of hydrogen into helium taking place only in a thin shell around the core. This process is called shell hydrogen burning.** See figure 4.10. This time core is helium rich and outer layer is hydrogen rich. The shell where energy production occurs is very thin.

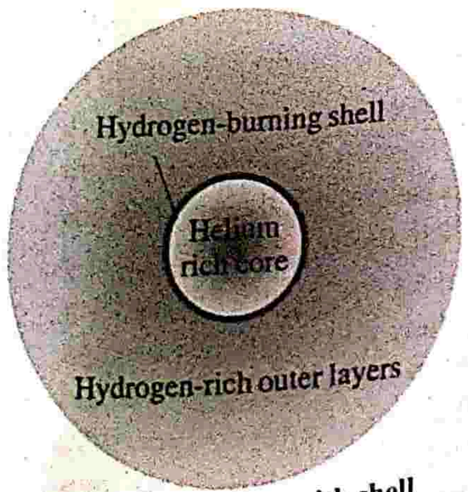


Figure 4.10: Star with shell hydrogen-burning

For a star like Sun, the hydrogen consuming shell develops immediately after the core ceases production of energy so that output energy is almost a constant. For massive stars the time interval to change from core nuclear fusion phase to the beginning of the shell hydrogen burning phase is of the order of thousand to million years.

The new supply of energy and thus heat further increase the rate of shell hydrogen burning. The by-product of the hydrogen fusion which is helium falls into the core which is already rich in helium. It heats up the core there by core again contracts and its mass increase. The core compression results in increase of temperature to about 15 million kelvin to 100 million kelvin.

At this temperature stars inner core is invisible to our eyes. Now we have a contracting inner core star with increased flow of heat and a ever expanding shell of hydrogen burning. As a result of this stars luminosity increases quite substantially. The increased temperature inner core increases the pressure, which makes the outer layer of the star to expand many times of their original radius. The tremendous expansion of the outer layer makes it cool. The temperature comes to about 3500 K. Thus the out layer seems to be in red colour. This giant red in colour star is called red giant.

There are so many red giants that are observable in the night sky. Some of them are Capella A, Arcturus, Aldebaren, Pollux, Mirah, etc.

According to the present knowledge most of the stars with a mass greater than or equal to the sun's will eventually become red giants. This means that one day our Sun becomes a red giant and large enough to swallow up mercury and venus.

Helium burning

When a star becomes a red giant the inner core is full of helium nuclei which is the byproduct of hydrogen burning nuclear reaction. Then helium will be next fuel for burning. This is the helium burning phase. As the star becomes a red giant the core temperature is too low to initiate helium burning. At the same time the hydrogen burning shell surrounding adds mass to the core, resulting in further contraction of the core. Owing to this core becomes denser and the temperature increases substantially. As the temperature increases the electrons in the gas become degenerate. This degenerate electrons resist further contraction of the core and the internal temperature can no longer affect the internal pressure.

As the hydrogen shell continues to burn, the degenerate core grows even hotter and the temperature becomes 100 million kelvin core helium burning begins, converting helium into carbon and oxygen releasing energy. During this state of star's life its size is only about 1AU in radius and luminosity is 1000 times brighter than the Sun. Once again, the old star which left the main sequence, obtained nuclear energy.

The helium burning in the core fuses three helium nuclei to form a carbon nuclei. This is called triple alpha process. This occurs in two steps.

Step 1

Two helium nuclei combine to form an isotope of beryllium.

i.e. $$4He + 4He \longrightarrow {}^{8}Be$$

This isotope of beryllium is very unstable and quickly breaks up into two helium nuclei. But in the extreme conditions in the core, a third helium nucleus fuses with beryllium before the break up and a stable isotope of carbon is formed and energy released in the form of γ-ray photon (γ)

i.e. $${}^{8}Be + {}^{4}He \longrightarrow {}^{12}C + \gamma$$

Since helium nucleus is also called as α-particle, the process is called triple alpha process.

Step 2

The carbon nuclei so formed can also combines with another helium nuclei producing stable isotope of oxygen and releases additional energy.

i.e. $${}^{12}C + {}^{4}He \longrightarrow {}^{16}O + \gamma$$

These byproducts (carbon and oxygen) of helium burning is called ash.

The formation of carbon and oxygen not only produces more energy but also re-establishes thermal equilibrium in the core of the star. The prevents the core from further contraction due to gravity. This life time of a red giant in the helium burning stage is only 20% as long as the life time it spend burning hydrogen burning on the main sequence. For example the life time of Sun in the main sequence is 10 billion years but it will spend only two billion years in the helium burning phase.

The helium flash

The mass of a star plays a vital role in deciding how helium burning begins in a red giant. If the mass of star is greater than 2-3$M_{\odot}$, the helium burning begins gradually as the temperature of the star approaches 100 million kelvin. Then the triple α process is initiated, but it occurs before the electrons become degenerated. **However if the mass of the star is less than 2-3$M_{\odot}$, the helium burning stage begins suddenly in a process called helium flash.** This is due to the most unusual conditions prevailing in the core of the star.

The energy that is produced by the triple α process heats up the core and its temperature rises. This increase and subsequent rise in energy production can cause the temperature to reach 300 million kelvin. It is due to the rapid heating of the core, a nearly explosive consumption of helium occurs. This is called helium flash. At the peak of the helium flash, the energy output of the core is about 10^{11} to 10^{14} times greater than the solar luminosity. When the temperature of the core becomes 300 million kelvin electrons become non-degenerate and they behave like ordinary electrons in a gas. The result is that the core expands which ends up in helium flash.

It may also be noted that the helium burning in the core lasts for a relatively short time. For example a star like Sun, the period after helium flash will only last about 100 million years which is about 1% of its main sequence life time.

Star clusters, red giants and the H-R diagram

So far we have been dealing with stellar evolution and its ultimate fate starting from protostar to red giant. During this very long processes, we noticed several changes regarding the temperatures, the life times, the luminosities, the phases etc. Now we are going to indicate all these changes in an H-R diagram, so we get evolution track of stars at one glance. This is depicted in the figure given below. Stars are formed from protostars and are about to join main sequence. That is we begin with a youngster star. The following points may be depicted on the H-R diagram.

1. A youngster star is in the main sequence and hydrogen burning is the process taking place and the star is in hydrostatic equilibrium. These stars are often referred to as zero age main sequence stars (ZAMS). This is represented by the curve I on the H-R diagram. From this curve we can read off the temperature and luminosity corresponding to mass of the star.
2. We know that life time of a star depends on its mass, larger mass stars lives shortly. During this life time hydrogen has been burning and converting into helium and the luminosity increases accompanied by an increase in stars radius. So the stars move away from curve I. Finally the stars reach at curve II. This curve represents the end of the hydrogen burning. That is main sequence star is spread over the space between curve I and II.

   This shows that the main sequence on the H-R diagram is a broad band or ribbon like rather than a single curve.
3. When hydrogen fuel has been exhausted the nuclear reaction ceases and there is a sudden fall in temperature. Thus the stars move from left (high temperature) to right (low temperature) on the H-R diagram. This is represented by evolutionary tracks with arrow marks. It may also be recalled that at this phase luminosity

remains almost constant. The core is contracting and outer layers expanding as energy flows from, the hydrogen turning shell (Red giant).

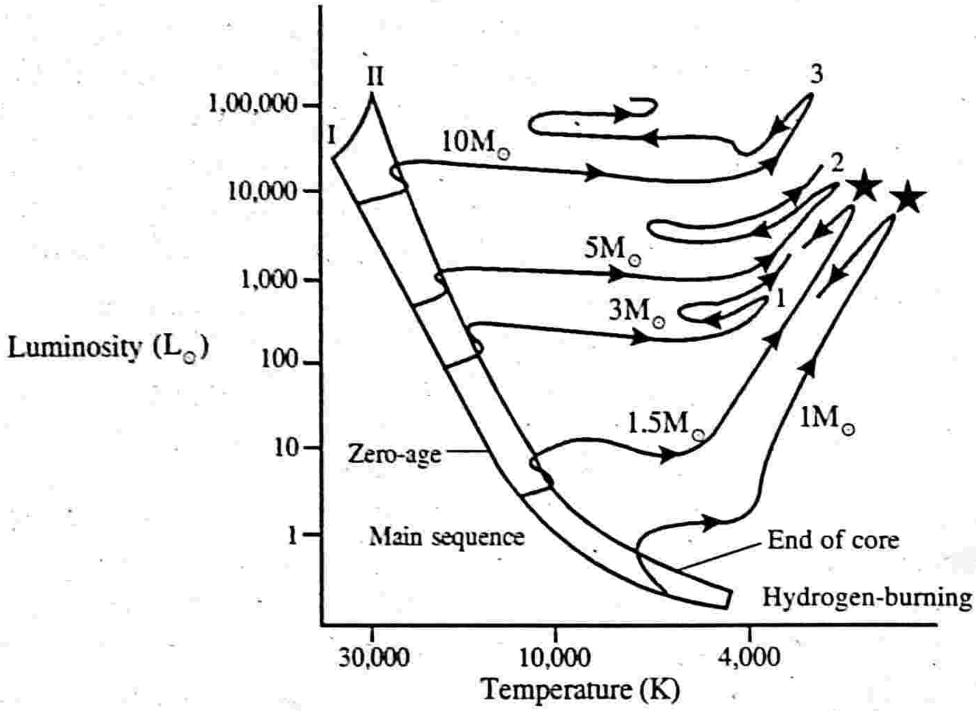


Figure 4.11: Post main sequence evolutionary track for several stars of different masses

4. The cusp of every evolutionary track represents a red giant. When the mass of red giant is small (less than 2-3$M_{\odot}$) helium flash occurs. This time the star shrinks and becomes less luminous and the temperature rises. So the evolutionary track moves down lower luminosity) and also to left (increase in temperature), on the H-R diagram. This is indicated by two stars symbols on the H-R diagram.

In the case of high mass stars (greater than 2-5$M_{\odot}$) core helium burning exhibit downward turns on the H-R diagram. See points 1, 2 and 3 marked. The evolutionary track then makes an upward turn to the upper right. This occurs just before the core helium burning begins. After the start of helium-burning, the core expands, the outer layers contract the evolutionary track fall from temporary high luminosities. It may also notice that how the truck moves back and forth on the H-R diagram. This represents the star's adjusting to their new energy supplies.

We can observe the evolution of stars from birth to the helium burning by looking youngster clusters and comparing actual observations with theoretical calculations.

Post main sequence star clusters: The globular clusters

A collection of gravitationally bound stars formed in spherical in shape and found around the galactic centre is called globular cluster.

A cluster contains about 10,000 to one million stars. Since the stars are metal poor they are found to be distributed in spherical shape around the galactic centre with a diameter about 10 *ly* to 300 *ly*. The number of globular clusters is found to increase as we move towards the galactic centre. Towards the galactic bulge there are high concentration of globular clusters. Sagittarius (Dhanu) and Scorpius (Vrichikam) are examples of globular clusters.

The origin and evolution of a globular cluster is very different from that of a galactic cluster. All stars in a globular clusters are very old. Any star massive than G or F-type might have already left the main sequence and moved towards the red giant stage. Thus no formation of new stars within the globular cluster is possible in our galaxy. Thus they are believed to be our galaxies oldest structure. In fact the youngest of globular clusters is still far older than the oldest galactic cluster.

Although the stars within a globular cluster are formed in similar conditions, stars differ in their sizes. However when we look at stars from the earth all seen to be

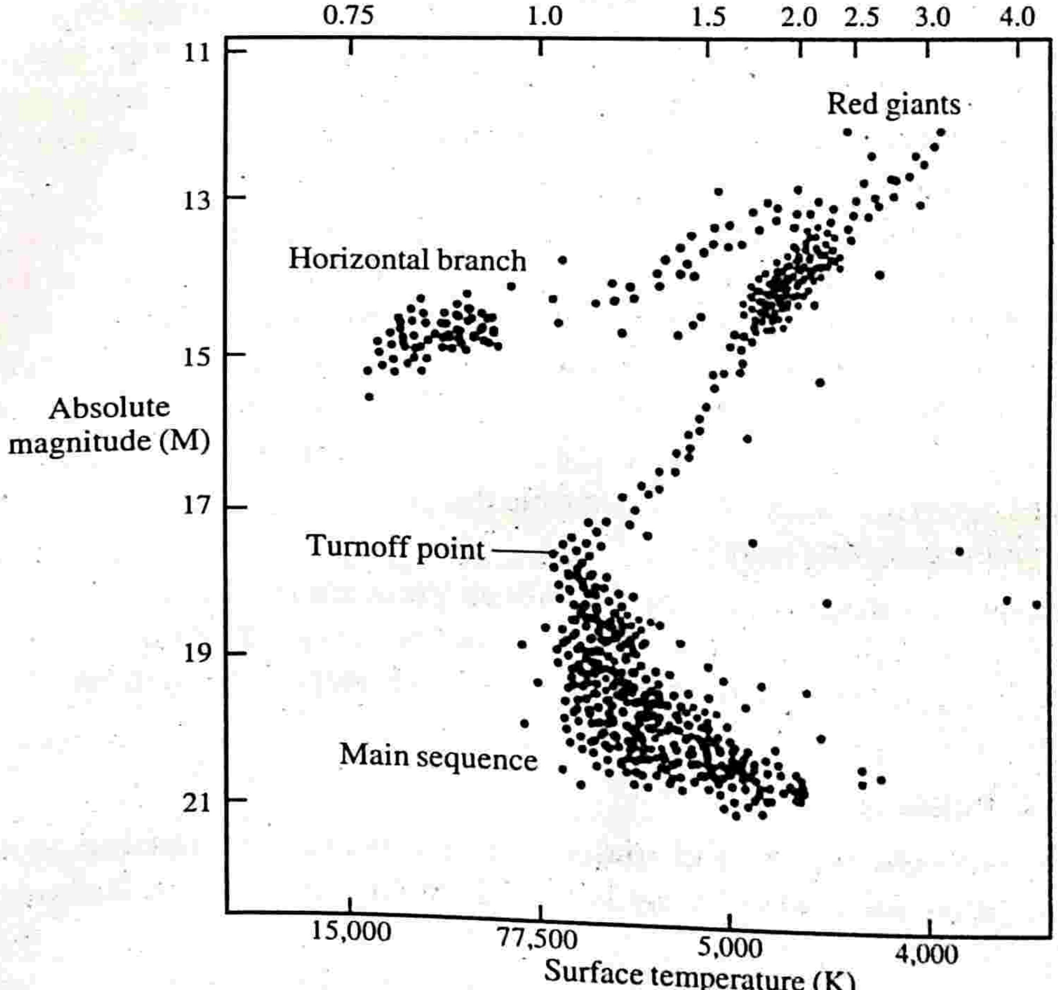


Figure 4.12: Colour-magnitude diagram for the globular cluster M3

about at the same distance. To find out the distance and age of a cluster we draw a special kind of H-R diagram called colour-magnitude diagram. On a colour-magnitude diagram, the apparent brightness is plotted against the colour ratio for many of the stars in a cluster. The colour ratio of a star can tell us the surface temperature of the star, when we compare colour-magnitude H-R diagram with an ordinary H-R diagram we can very well calculate the distance of a cluster. If the stars in a cluster lie at the same distance the apparent brightness (magnitudes) can tell us their relative luminosities. A colour-magnitude diagram for the globular cluster M3 is given figure 4.12.

The following informations are obtained from the colour-magnitude diagram.

1. In the graph we can seen that the upper half of the main sequence has disappeared. This means that all of the high mass stars in a globular cluster have evolved into red giants, a long time ago. The low-mass main sequence stars (lower part of the graph) are very slowly turning into red giants.
2. On the left of the diagram (up) we can see grouping of stars. This is called the horizontal branch. Stars in this horizontal branch are called horizontal branch stars. These stars are low mass, post helium flash stars with luminosity about $50L_{\odot}$. In these stars there are both core helium burning and shell hydrogen burning. In the future these stars will also move towards the red giant region as the fuel is exhausted.
3. As time passes, the main sequence stars will be smaller and smaller in number. The top of the main sequence, which remains after the specified time can be used to determine the cluster's age and is called the turnoff point. The stars that are at the turnoff points are those that are just exhausting the hydrogen in their cores, so the main sequence lifetime is in fact the age of the star cluster.

Many stars in the globular cluster are visible to optical instruments. Some are even visible to the naked eye. Some stars not visible due to the presence of gases and dust particles near the galactic centre.

The nearest globular cluster, for example Caldwell 86 in ara (althara) constellation lie at a distance of 6000 ly from the galactic plane. So it cannot be detected by small telescopes. Even the brightest and biggest globular cluster can be seen only by telescope of aperture 15 cm.

Pulsating stars (Pulsars)

Stars which emit light signals and undergoing continuous expansion and contraction at regular intervals of time is called pulsating variable stars or shortly pulsars.

Stars which emit radiosignals are called radio pulsars, those which give out X-rays are called X-ray pulsars. It has also been discovered that some stars emit γ-ray pulsars.

First pulsar was detected in the year 1967 by Joselyn Bell and Anthony Hewish. This is named as PSR 1919 + 21. The time period of the pulse emitted by this pulsar is 1.337 seconds. For this discovery Anthony Hewish was awarded the physics Nobel Prize in the year 1974. In those days people believed that the signals are sent by some intelligent extraterrestrial creatures. But 1978 Thomas Gold and Franco Pachini suggested pulsars are rotating neutron stars.

The time period of the pulse is not the same for all pulsars. It varies from some milliseconds to 5 seconds. For example the time period of Crab pulsar is 0.033 second. It is due to the high precision of time period the pulsars are regarded as the cosmic clocks.

On the basis of the H-R diagram we can explain pulsars. We found that when star's masses are far massive than the Sun that contract and move horizontally across the H-R diagram, while at the same time they get hotter but remain at a constant luminosity. As they move across the H-R diagram these stars are unstable and change their size by alternately by contracting and expanding.

There are several classes of pulsars such as long period variables, the Cepheid variables, and RR Lyrae variables. These will be discussed later. The figure below shows where these different classes of pulsars reside on the H-R diagram.

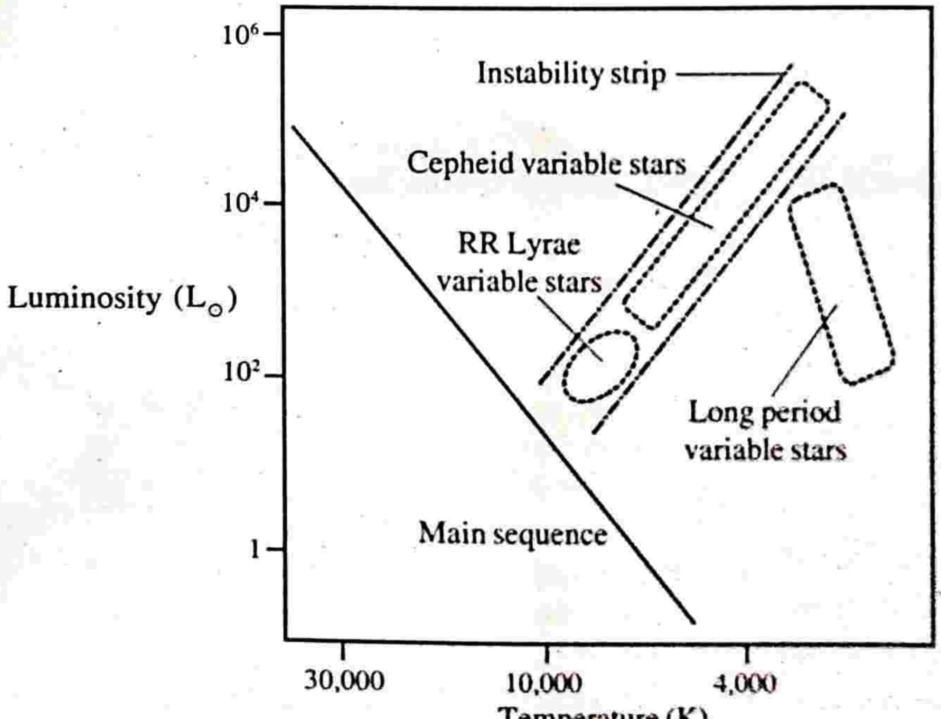


Figure 4.13: Variable stars on the H-R diagram

Why do stars pulsate

It has been realised that some stars pulsate not due to the variations in the rate of energy production in the inner core but due to changes in the rate at which energy can escape from the star. The explanation is as follows.

Imagine a normal star, where there is a perfect balance between the gravitational force acting inward and the force due to radiation pressure acting outward. Now suppose that somehow the force due to radiation pressure of the star at the out layers exceeds the gravitational force of the outer layers the star would begin to expand at the outer layers. As the star expands naturally gravity force falls down $\left(F \propto \frac{1}{r^2}\right)$, but the pressure force will fall at a faster rate. A time would then come when the star will have expanded to a large size such that hydrostatic equilibrium is once again regained. This does not mean that star would stop expanding. It is due to inertia expansion continues, thereby upsetting the balance. But the gravity pull will stop the expansion as the radiation force is too low to balance the gravity. So the outer layers would begin to fall inward. At this point gravity will rise again $\left(F \propto \frac{1}{r^2}\right)$ but less than the pressure. The outer layers will fall past the balance point until the force of pressure would prevent any further fall, so would come to a halt. The process continues forever. As a result star pulsates. See figure 4.14.

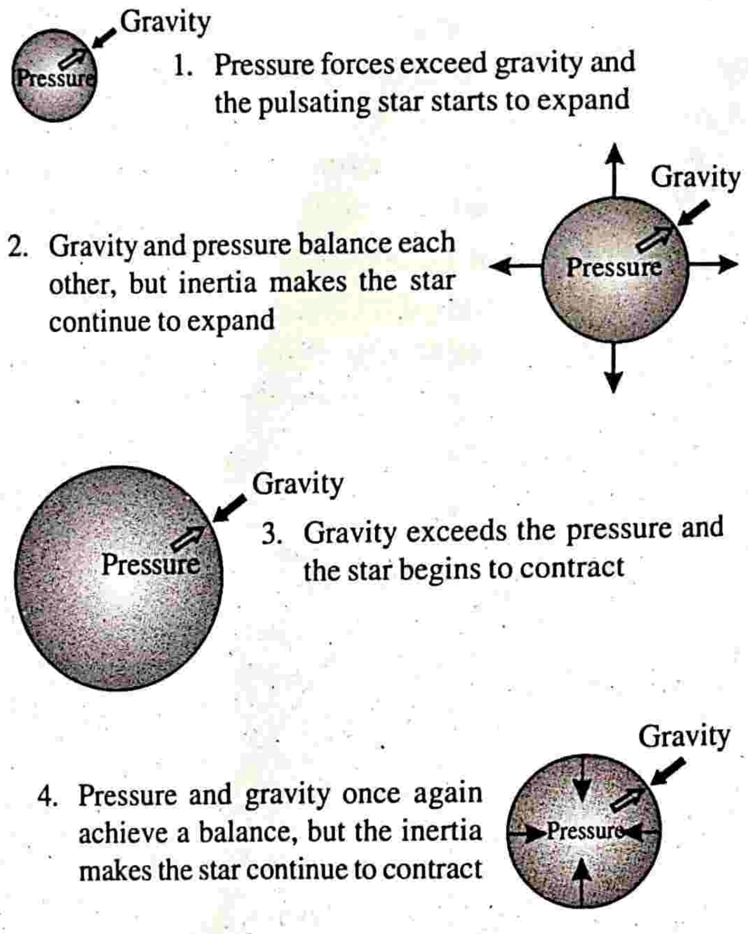


Figure 4.14: Gravity and pressure during the pulsation cycle of a pulsating star

The pulsating star behaves something like the oscillation of a spring attached to a mass. In this case there is a balancing and unbalancing between gravity and tension acting on the spring. After sometime the oscillation will die due to the presence of damping force. In order to sustain the oscillation it requires outward push. In the case of pulsars what is the extra force that would sustain the expansion and contracting was a challenging problem to astronomers.

In 1914 the British astronomer Aurther Eddington gave an explanation for this. He suggested that a star pulsated because its opacity increases more when the gas is compressed than when it is expanded. Heat is trapped in the outer layers if a star is compressed, which increases the internal pressure this in turn pushes the outer layers. As the star expands the heat will escape and so the internal pressure falls and the stars surface drops inward.

In 1960, the American astronomer Johncox further developed the idea of Eddington and proved that helium is the key to the pulsation of stars. When a star contracts, the gas beneath its surface gets hotter, but the extra heat does not raise the temperature instead, it ionises the helium. This ionised helium is very good at absorbing radiation. In other words, it becomes more opaque and absorbs radiant energy flowing outward through it. This trapped heat makes the star expand. This then provides the push that propels the surface layers outward. As the star expands, electrons and helium ions recombine and this causes the gas to become more transparent.

In effect we can say that for sustained pulsations star must have a layer beneath its surface in which helium is ionised. The existence of such a layer depends on the size and mass of the star as well as on the temperature. The range of this temperature is from 5000 to 8000 K. There is a region on the H-R diagram, where such an area exists and it is the location of the pulsars. This is called the instability strip. The Cepheid variable pulsars and RR Lyrae pulsars are found in this region.

Cepheid variables and the period-luminosity relationship

Cepheus is an important and noteable constellation in the northern celestial sphere. It lies between Cassiopeia and Draco (Vyali) constellations. According to Greek mythology the constellation is the replica of king Cepheus of Ethiopia, hence the name Cepheus constellation. This constellation contains so many pulsars such as delta cephei, mucephei, VV cep, lambda cephei etc. δ- cephei is a pulsar with periodicity 5.4 seconds. It is a yellow giant star whose luminosity changes by a factor of two over 5.4 seconds. It has a companion star with blue colour. This means that it is a binary star. Stars which contract and expand with a constant periodity are called are Cepheid variables. Among these groups of stars the first star to be discovered was δ- cephei variable star. **All stars discovered after δ- cephei were called as Cepheid variables.** Variation of δ- cephei in luminosity, size and temperature versus time period are plotted in the same graph.

From the graph it can be seen that its luminosity and temperature have a maximum value when its size has a minimum value and vice-versa. That is its size is at its maximum when its luminosity and temperature are at minimum.

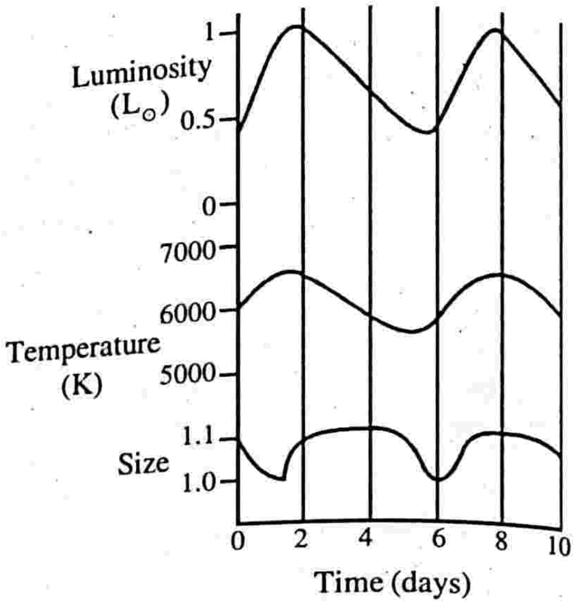


Figure 4.15: The size, temperature and luminosity of δ- cephei during one period

The study of Cepheids are very important for two reasons. (1) The Cepheids are very luminous ($100L_\odot$ to $10{,}000L_\odot$) they can be seen even though they are at great distances (few million parsecs) (2) There exists a relationship between the pulsation period and its average luminosity. This is called period-luminosity relationship.

If a star can be identified as Cephied its period can be measured. Then its luminosity and its and its absolute magnitude can be determined. This can then be used along with its apparent magnitude to determine its distance.

Cepheid stars are classified into two according to the amount of metals present in the stars outer layers which a determines how it pulsates. This is because metals can have substational effect on the opacity of the gas. If a cepheid is a metal rich population I star it is called a type I Cepheid and if it is a metal-poor population II star, it is called type II Cepheid. Period-luminosity relationship for the two types of cepheids are shown in figure (4.16).

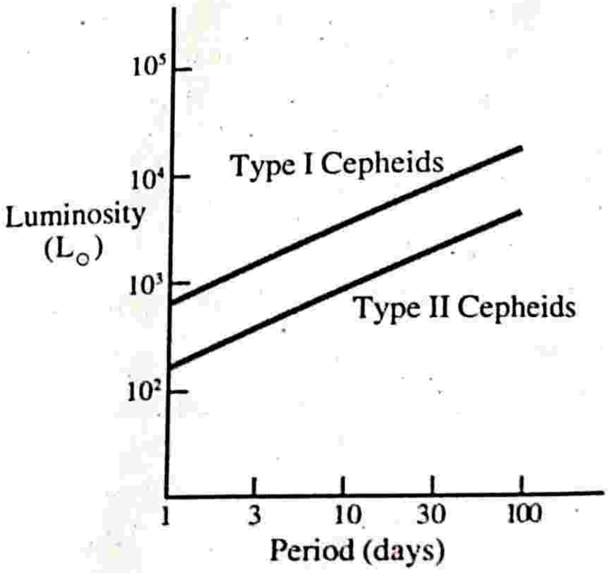


Figure 4.16: Period-luminosity relationshp for the two types of Cepheid variable star

Note

Population I star

These are bright, supergiant, main sequence stars with high luminosity. Examples O-type, B-type and members of galactic clusters and δ- cephei.

Population II star

These are old stars found in globular clusters. These are also called as ω-Virgins.

Examples: RR Lyrae and the central stars of planetary nebulae.

Temperature and mass of Cepheids

The period luminosity relationship comes about because the more massive stars are also the most luminous stars as they cross the H-R diagram during core helium burning. These massive stars are also larger in size and lower in density during this period of core-helium burning and the period with which a star pulsates is larger for lower densities. So the massive pulsating stars have greater luminosities and longer periods. This is shown in figure below.

We have seen that old high mass stars have evolutionary tracks that cross back and forth in the H-R diagram and thus will intercept the upper and of the instability strip. Such stars become cepheids when the helium ionises at just the right depth to drive the pulsations. Those stars on the left (high temperature) of the instability strip will have helium ionisation occurring too close to the surface and involve only a small fraction of stars mass. The stars on the right (low temperature) side will have convection in the stars outer layers and this will prevent the storage of the heat necessary to drive the pulsations. Thus Cepheid variable stars can only exist in a very narrow temperature range.

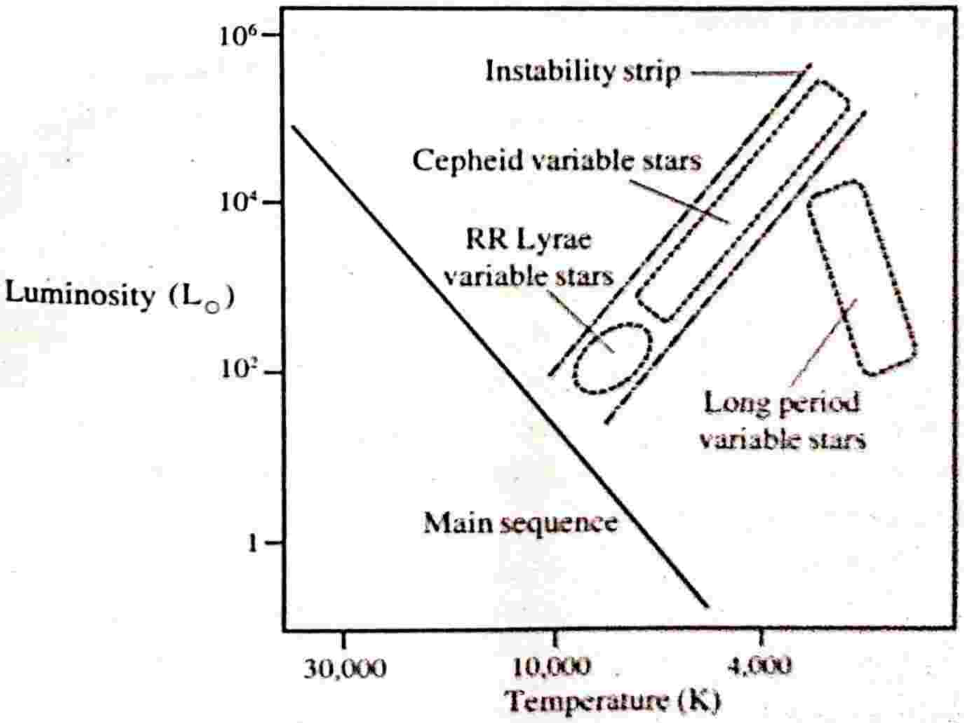


Figure 4.17: Instability strip and evolutionary tracks for stars of different mass

The death of stars

So far we learned about the formation of protostar, pre-main sequence stage, main sequence stage and red giant. It is the mass of a star decides how it will end its life. Low mass stars can end their lives in planetary nebulae before proceeding to white dwarf stars, where as high mass stars end their lives in supernova explosion. Depending upon the mass of the remants of supernova explosion it may become a neutron star or a blackhole. We shall see this one by one.

Now one very important question to be addressed stars live for millions, billions or even hundreds of billions of years, then how can we say affirmatively that a star dies? After all, we have only been in this planet for 4.5 billion years and studying astronomy for about 10,000 years. Fortunately, nevertheless it is possible to observe the many fundamentally different ways in which a star can end its life.

Asymptotic giant branch

We found that how a star becomes a red giant at the end of its main sequence life time. When a star becomes a red giant it moves left across the H-R diagram along the horizontal path as its luminosity remains almost constant. This is called the horizontal branch. When the star (Red giant) is in the horizontal branch it has helium burning core which is surrounded by a shell of hydrogen burning. It is due to α-triple process carbon and oxygen are the by-products present in the core. This will continue for a long period time about 100 million years. During this time all helium fuel has been used up and converted into carbon and oxygen when helium fuel exhaust nuclear fusion stops. Thus force of radiation is not able to balance the gravity force so the core contracts. However the core contraction is stopped by the degenerate electron pressure. The result of core contraction is the release of heat by the core into the helium gas surrounding the core. So helium burning begins in a thin shell around the carbon-oxygen core. This is called shell helium burning. Now the star enters a second red giant phase. **The hydrogen shell burning causes the outer layers to expand and cool. The energy from the helium burning star also causes the outer layers to expand and cool. So the low mass star rises into the red-giant region of the H-R diagram for a second time. This time red giant has greater luminosity. This phase of a star's life is called the asymptotic giant branch phase or AGB. The stars in this region are called AGB stars.**

The AGB star consists of a central core with carbon-oxygen mix surrounded by a helium burning shell which in turn is surrounded by a helium rich shell. This is further surrounded by a hydrogen burning shell.

Now the size of the star is gigantic. The core region is about the same size as the earth and total size is as large as the orbit of the earth. The luminosity of these stars are very high. For example an AGB star of mass $1M_{\odot}$ its luminosity is about $10,000L_{\odot}$. It is calculated that after 8 billion years our Sun becomes an AGB star.

There are so many AGB stars have been discovered. Some of them are star mira belongs to Ocetius (Thimingalam) constellation, R leonis belongs to Leo (Simham) constellation, R aquarri belongs to Aquaris (Kumbam) constellation.

The pictorial representation of the structure of an AGB star is given below.

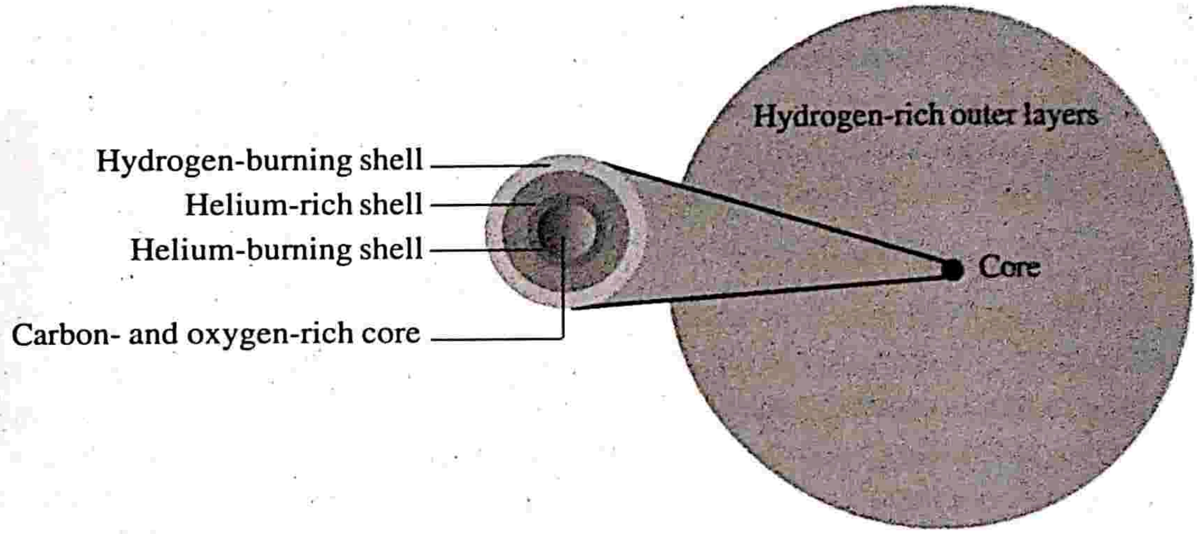


Figure 4.18: The structure of an AGB star

The end of AGB star's life

A red giant in phase I after about 100 million years reaches in phase II region called an AGB star. As time passes an AGB star grows in size and increase in its luminosity along with an increase in the rate of loss of mass. The mass loss can be $10^{-4}M_{\odot}$ per year. This means that our Sun after becoming an AGB star lives only for another 10,000 years. This shows AGB stars cannot go long away. If a star of mass less than $8M_{\odot}$ its stellar wind will strip away the outer layers almost down the core. This indicates the end of the AGB star. For stars greater $8M_{\odot}$, AGB star will end up in an explosion called supernova. This will be discussed later.

The formation of AGB star results in the production of carbon and oxygen via α-triple process. This will enrich the interstellar medium with carbon and oxygen. This indicates that the carbon in our body and all living creature on earth was formed many billions of years ago inside an AGB star by the α-triple process. It was then dredged up to the stars surface and expelled into space. Later by some means, it formed the precursor to the solar system and planets and all life on earth. This leads us to think that everything made of the stuff of stars.

Planetary nebulae

The interstellar space which consists of gases and dust particles of clouds is called as nebula. Nebula is a Latin word meaning cloud. In the study of stellar evolution nebulae plays an importance since nebulae are the birth places of stars. The debris of star explosion also come under the definition of nebula. Here we discuss about planetary nebulae.

A planetary nebula is an interstellar space containing gases and dust particles of clouds, fastly moving debris thrown away by the AGB stars at their ends of life with glowing core of stars. The planetary nebula was firstly discovered by astronomer Charles Messier in the year 1764. But the name was given by the astronomer William Herschel in the year 1783.

Formation of planetary nebulae

At the end of the AGB phase what remains is the degenerate core of carbon and oxygen surrounded by a thin shell in which hydrogen burning occurs. The dust ejected in AGB phase will be moving outward at tens of kilometres per second. As the debris moves away, the hot, dense and small core of the star will become visible. The aging star will undergo a series of bursts. During each burst ejects a shell of material into the interstellar space. The glowing inner core gives out almost constant luminosity for few thousand years and temperature of this comes about 30,000K to 100,000K. At this temperature star will emit huge amount of ultraviolet radiations, which can excite and ionise the expanding shell of gas. The totallity of all these is called as planetary nebula.

In our galaxy alone there are about 1500 planetary nebulae were discovered so far.

White dwarf stars

We found that how AGB stars end their lifes. AGB stars which have masses less than $4M_{\odot}$, the internal pressure and temperature produced inside the cores at its end of their life cannot ignite the carbon and oxygen in the cores. In this situation the stars throw away their outer layers leaving behind very hot carbon-oxygen rich cores. So the cores stopped their energy production by nuclear fusion and begins to cool down over a vast times scale. The cooling dead bodies or remnants of stars are called white dwarfs. As these stars emit white light and smaller in size they are called white dwarfs. The size of such stars are about 300 kilometres only.

Electron degeneracy

As the star sheds away its outer layer, it begins to contract. As the size decreases the mass of the white dwarf increases so the gravity pulls increases. The increase in mass and decrease in size is unlikely to our thinking. But this is true for white dwarfs. This is because of high density increase. The temperature that exists in the core brings the electrons to a peculiar state of affair called degeneracy. The degeneracy is a quantum phenomenon. According to Paulis exclusion principle no two electrons can occupy the same quantum state, so electrons progressively go to higher energy states to occupy. As a result they acquire high velocity and pressure. This pressure is called degenerate pressure. The force of degenerate pressure balances the gravity pull of the contracting star. This is what is happening inside a white dwarf. It is further emphasised that larger the white dwarf's mass smaller its size.

The Chandrasekhar limit

It is due to the degeneracy of matter inside the core, the density of the star increases so also its mass. The increase in mass results in increase gravitational pull, so the size of the star decreases. This means that when more and more degenerate matter is formed the size of the star becomes smaller and smaller. However this increase in mass cannot go indefinitely. **It has been calculated by the Indian astrophysicist Subramanyam Chandrasekhar that there is a maximum limit to the mass attained by white dwarfs. The mass is about $1.4M_\odot$. This limiting mass of white dwarfs is called Chandrasekhar limit.** For this discovery Subramanyam Chandrasekhar was awarded the physics Nobel Prize in the year 1983. When the mass of a star exceeds this mass limit $1.4M_\odot$, the force due to degenerate radiation pressure cannot balance the gravity pull. As a result star again contracts thereby electrons and protons inside the core fuses to form neutrons. Depending upon the mass of the star it may become a neutron star or a blackhole.

The mass-radius relationship of white dwarf stars are depicted in the figure 4.19.

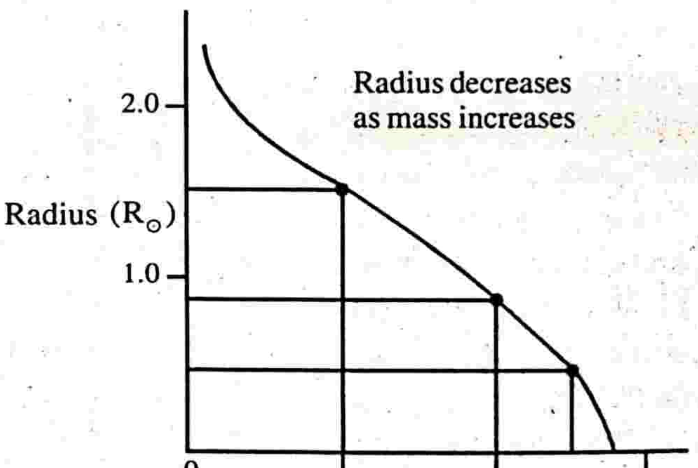


Figure 4.19: Mass-radius relationship for white dwarf stars

Note: The mass of the star is expressed in terms of $M_\odot$ and radius is expressed in terms of that of earth ($M_\odot$).

Which stars become white dwarfs in their old age. The stars whose mass is less than $1.4M_\odot$ in their old age become white dwarfs. The star of types O, B and A which have larger masses can also become white dwarfs provided, they shed of their masses during their AGB phase II. After shed of mass if the mass is less than $1.4M_{\odot}$ become white dwarf. On the other hand they become neutron stars or black holes.

The final question to be answered is "what is white dwarf star made of"? It consists of mainly ionised oxygen and carbon atoms along with fast moving degenerate electrons. As the star cools down the electric forces between ions dominate over their random thermal motion. So ions no longer move freely but are aligned in orderly fashion and they behave like a giant crystal lattice. In this crystal lattice the degenerate electrons move freely in this giant crystal like electrons move freely in a copper wire. The density of white dwarf is about 10^9 kgm^{-3}. This is about one million times the density of water (10^3 kgm^{-3}). For example a teaspoon of white dwarf weights about 5.5 tonnes equal to the weight of an element.

$$M = V\rho, \quad V = 5.5\text{cm}^3 = 5.5\times10^{-6}\text{m}^3$$

$$\therefore \quad M = 5.5\times10^{-6}\times10^9 = 5.5\times10^3\text{kg}$$

$$M = 5.5 \text{ tonnes.}$$

Volume of the teaspoon is 5.5 cm^3 on the average.

White dwarf evolution

When an AGB phase II star becomes a white dwarf it shrinks to an ultimate size with no nuclear fuel. However it will still have a very hot core with immense heat so the surface temperature is very high. For example the surface temperature of white dwarf Sirius B (the companion of Sirius A) is about 30,000K. As time passes heat will be radiated into space and they become cool, so their luminosity become lesser. The more massive white dwarfs will have smaller surface are so also their luminosity, for a given temperature. When we represent this

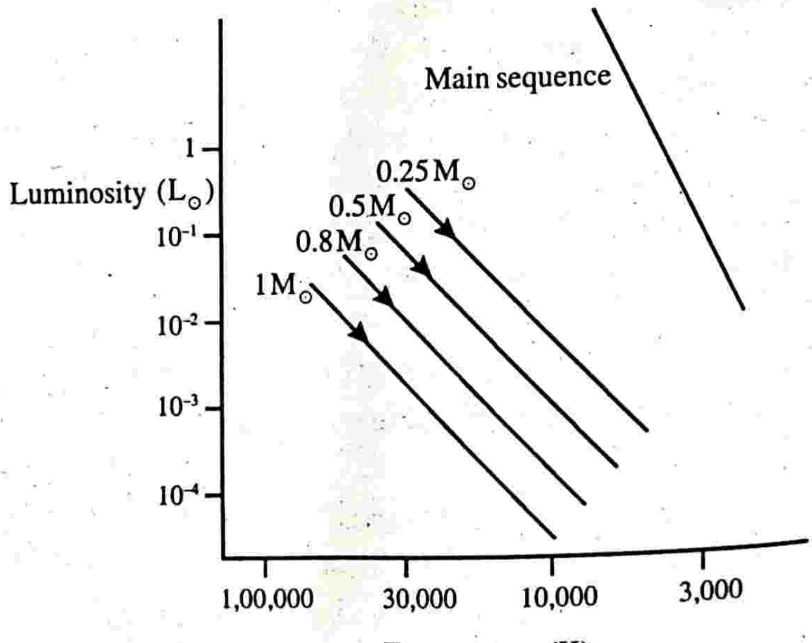


Figure 4.20: White dwarf evolutionary tracks

on a H-R diagram the evolution tracks of massive stars are below those of the less-massive stars. An H-R diagram showing the evolutionary tracks of white dwarf stars with different masses are shown Figure 4.20.

The theoretical model of the evolution of white dwarfs gives us important informations. The white dwarfs with a mass of $0.6M_\odot$ will fade to $0.1L_\odot$ in about 20 million years. Any further reduction in luminosity takes longer times. This means that it will take 300 million years to fade to about $0.01L_\odot$ and billion years to take to get to $0.001L_\odot$. It will take about 6 billion years to reach about $0.0001L_\odot$. At this luminosity white dwarf will have the same temperature and colour (white) of the Sun. The white dwarfs with masses greater than $0.6M_\odot$ have more heat content and so will take even longer time to cool down and grow faint.

In the case of the sun, it will eject most of mass into space and eventually ends up about the same size as the earth, but luminosity changes dramatically and it becomes $0.01L_\odot$. After 5 billion years the luminosity becomes $10^{-5}L_\odot$ and gradually fades away from our view.

The white dwarf origins

All, so far, discovered white dwarfs are found to be originated from the central part of the planetary nebulae where AGB II phase stars were living and end their lifes. The masses of stars were also found in well agreement with Chandrasekhar limit ($1.4M_\odot$). But even though theory matches with experimental observations, there is still uncertainty in the initial mass of the stars leading to white dwarfs. Current ideas suggest a limit of $8M_\odot$. Those main sequence stars that have between 2 and $8M_\odot$ produce white dwarfs of mass of 0.7 and $1.4M_\odot$, where as main sequence star less than $2M_\odot$ produce white dwarfs of mass 0.6 to $0.7M_\odot$. The lower mass stars in the main sequence have incredibly long lifetimes, the universe is not old enough to produce white dwarfs. From this we can conclude that there are no white dwarfs with mass less than $0.6M_\odot$. The time taken for the evolution from giant stars to white dwarf can be in between 10,000 to 100,000 years.

Some examples of white dwarf stars are Sirius B belongs to Canis major (big dog), Procyon B belongs to Canis minor (small dog), 0 eradani 40 belongs to Eradinus (river) and Van Maneen's star Wolf 83 belong to Pisces (fish) constellations.

High mass stars and nuclear burning

The life of high-mass stars are different from these of low-mass stars. Throughout the entire mass of a low-mass star only two nuclear reaction occur. The hydrogen burning and helium burning. The by-products of this burning are carbon and oxygen.

In the case of zero age mass greater than $4M_\odot$, the temperature involved is very high so several other nuclear reactions will also occur. Since the carbon-oxygen core is more massive than the Chandrasekhar limit $1.4M_\odot$ the gravity pull is very high and so the degenerate pressure cannot stop the core from contraction and heating.

After hydrogen burning and helium burning some helium will be left in the core. This cannot initiate further nuclear reaction. But the process of helium capture occurs. This is the process of fusing of helium into progressively heavier elements. The core begins to contract with rise in temperature to about 600 million kelvin. At this high temperature, the helium capture can give rise to carbon burning and the carbon can be fused into heavier elements such as oxygen, neon, sodium and magnesium. The carbon fusion produces a new source of energy which restores the balance between pressure and gravity temporarily.

If the star has mass greater than $8M_\odot$ further reactions will occur. In this phase, the carbon burning may only last a few hundred years. The core contracts further and temperature rises. When the temperature reaches 1 billion kelvin, the neon burning begins. Neon is the by-product of earlier carbon burning reaction. In neon burning there is an increase in the amount of oxygen and magnesium in the core. The neon reaction lasts as little as one year. In each stage of reaction temperature increases there by further reactions occur. When the temperature reaches 1.5 million kelvin oxygen burning will occur with the production of sulphur. When the temperature reaches to 2.7 million kelvin silicon burning occurs. This reaction produces several nuclei from sulphur to iron.

Despite the very dramatic events that are occurring inside the high-mass star its outward appearance changes only slowly. When each stage of core nuclear reaction stops, the surrounding shell burning intensifies and therefore inflates the star's outer layers. Then each time the core flares up again and begins further reactions, the outer layers may contract slightly. This is the reason why the evolutionary track of high mass stars moves in zig-zag path. See figure 4.11.

Some of the reactions that occur also release neutrons. They collide with positive charged ions and combine with them. The absorption of neutrons by nuclei is termed neutron capture. In this reaction many elements and isotopes are produced.

It is due to stars high mass events occur at a very fast rate, with each successive stage of nuclear burning proceeding at an ever increasing rate. Calculations show that for 20-25$M_\odot$ zero age stars the carbon burning stage can last for about 600 years, while neon burning stage can be as short as one year. The oxygen burning last only 6 months and the silicon burning only one day.

At each core burning, a new shell of material is formed around the core of the high mass star and after several such stages the internal structure of the star resembles an onion. See figure below.

Nuclear reactions are taking place in several different shells simultaneously and the energy released will heat a rapid rate such that the out layers can expand to huge size. The star now is called as supergiant. The luminosity and temperature will be very much higher than those of giant stars.

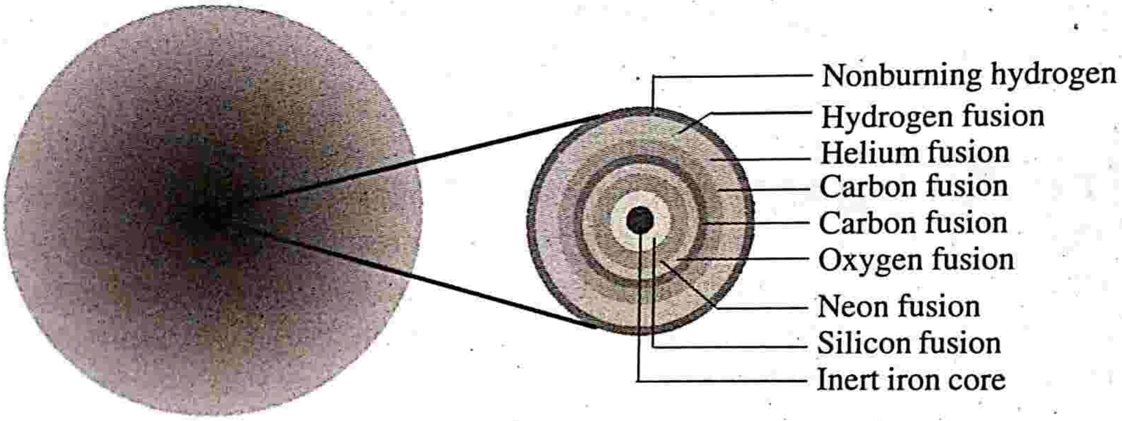


Figure 4.21: The multiple-layer structure of an old high-mass star 20 - 25$M_\odot$

Many of the brightest stars in the night sky are supergiants. Some examples are

1. Rigel and Betelgeuse in Orion (hunter) constellation.
2. Arcturus in Scorpius (scorpion) constellation
3. V.V. Cephi in Cepheus constellation

The size, temperature of the above supergiant stars are given below.

Name	Temperature in K	Size $M_\odot$
Rigel	11,000	——
Betelgeuse	3700	700
Arcturus	——	——
V.V Cephi	——	1990

Supernovae and formation of the elements

When a massive star becomes a supergiant, it cannot go on forever as such as there is only finite material to burn. Thus the star undergoes yet another change (gravitational collapse). It is star death. The change is highly catastrophic at the same time spectacular. This gravitational collapse of the star is called supernova.

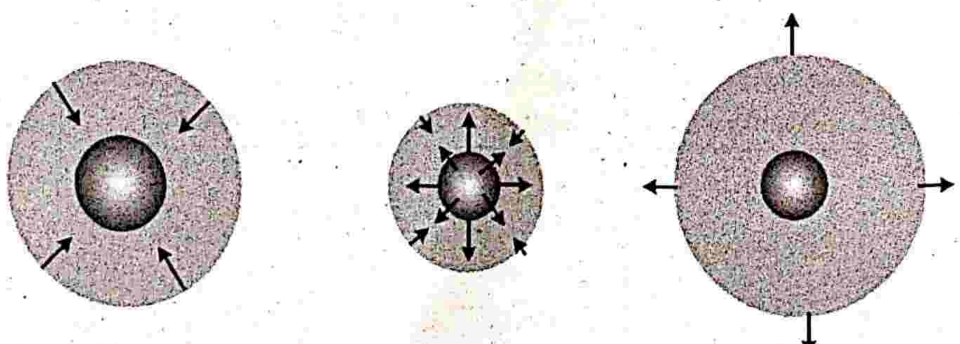
During the final days of supergiant, the core of inert iron, in which there are no nuclear reactions taking place, is surrounded by shells of silicon, oxygen, neon, carbon, helium and hydrogen. Since the mass is very heavy the force due to degenerate pressure cannot balance the gravity force. Hence the star undergoes collapse.

A consequence of core contraction is an increase in density, which in turn gives rise to a process called neutronisation. This is a process in which electrons react with protons in iron nuclei to form neutrons and neutrinos.

i.e. $$e^- + p \longrightarrow n + \upsilon$$

This results in speeding up of contraction (collapse). In seconds the core of radius of thousands of kilometres to about 50 kilometres. Then in few seconds it further contracts to 5 km. This time core temperature increases to about 500 million kelvin. The gravitational energy released as a result of the core collapse is equal to Sun's luminosity for several billion years. Most of this energy is in the form of neutrinos, but some is in the form of gamma rays, which are created due to the extremely hot core temperature. These gamma ray photons interact with iron nuclei resulting in the production of alpha particles. This process is called photodisintegration.

After a very short time of about 0.25 seconds, the core of mass of about $0.6M_\odot$ to $0.8M_\odot$ will reach a density equal to that of nuclei about $4\times10^{27}\,kgm^{-3}$. At this point neutrons become degenerate and force due to degenerate pressure balances gravitational pull and further contraction stops immediately and the inner most part



Step 1: The iron core collapses Step 2: The neutron-rich core rebounds Step 3: The shock wave moves outward

Figure 4.22: Evalutioin of a supernova explosion

of the core becomes rigid. This called a neutron star. The inner most part actually rebounds outward and pushes back against the rest of the infalling core, driving it outward in a pressure wave. This is called the core bounce. This is illustrated in figure 4.22.

At this stage the core cools down so also the pressure. The result is that the balancing will be upset. Owing to this the material surrounding the core falls inward at a speed close to 15% of the speed of light. This inward moving material encounters the outward moving pressure wave which moves at a speed of one sixth the speed of light. In just a fraction of a second the falling material now moves back outward towards the stars surface.

The upward moving wave of pressure speeds up as it encounters the less dense regions of the star and achieves a speed greater than the speed of sound wave. The pressure wave now becomes a shock wave. The neutrinos present in the core escape from the star in a few seconds but the shockwave takes few hours to reach the surface. Most of the material of the star is pushed outward by the shockwave and is expelled from the star at many thousands of kilometres per second. The energy released during this event is about 10^{49} J, which is 100 times more than the entire output of the Sun during the last 4.6 billion years. The visible light that we see during the event is only 1% of the energy released.

It has been proposed by astrophysicists that up to 96% material making up the star may be ejected into stellar medium that will be used in future generations of star formation. But before the matter is ejected, it is compressed to such a degree that new nuclear reactions can occur within it. It is these reactions that form all the elements that are heavier than iron. Elements such as tin, zinc, gold, mercury, lead, uranium etc. are produced. These elements that make up the solar system, the earth was formed long ago in a supernova.

From statistical considerations it is calculated that there should be about 100 supernovas in a year in our galaxy alone. In the year 1987 we observed a supernova in the large megallance cloud galaxy another one observed in our galaxy several hundred years ago. Since galaxy is filled with dust and gas they block the light coming from supernova, so it is very difficult to observe them. Now we believe that the stars Betelgeuse and Eta carina are on the verge of becoming supernovae.

Supernova remanants

Supernova remanant (SNR) is the totality that includes the debris of the explosion, the layers of the star that have been thrown into space and the remains of the core which is a neutron star.

The visibility of SNR depends on several factors such as its age, energy source that remains to make it shine and the type of supernova explosion.

As the remnant ages, its velocity will decrease from 10,000 kms^{-1} to about 200 kms^{-1}. During this time it will fade. A few SNRs have a neutron star at their centre that provides a replenishing source of energy to the far-flung material.

An example of SNR that undergoes this process is the crab nebula MI in Taurus (buffalo) constellation. What we see is the radiation produced by electrons travelling at velocities near the speed of light as they circle around the magnetic field. This radiation is called synchrotron radiation. This radiation is pearly, faint glow that we observe. Some SNRs glow as the speeding material interacts with dust grains and atoms in interstellar space, while others emit radiation due to the tremendous kinetic energies of the exploding material.

Caldwell 33 and 34 belongs Cygnus (swan) constellation, Messier I belongs to Taurus, sharpless 2-276 belongs to Orion (hunter) constellation are examples of supernova remanants.

Supernova types

Supernovae can be classified into two types depending on the spectra emitted by them. They are (i) type I supernovae and (ii) type II supernovae.

Type I supernovae

Supernovae which contain no emission lines of hydrogen in their spectra are called type I supernovae.

Type I can be further divided into types type Ia, type Ib and type Ic. Type Ia has absorption lines of ionised silicon and types Ib and Ic do not. Type Ib has helium absorption line whereas type Ic does not.

Type II supernovae

Supernovae which contain emission lines of hydrogen in their spectra are called type II supernovae.

Distinctions between various supernovae

(i) Types Ib, Ic and II are massive stars but type Ia stars have had their outer layers stripped away either by a strong stellar wind or action of a nearby star.

(ii) Type Ib, Ic and II are found near sites of star formation since massive stars have short lives. But type Ia supernovae are found in galaxies where star formation may be minimal or has even stopped altogether.

(iii) Type I supernovae involve nuclear energy and emit more energy in the form of electromagnetic radiation, whereas type II involve gravitational energy and emit enormous number of neutrinos.

Pulsars and neutron stars

We found that neutron stars are created at the middle of supernovae renanants. When a massive supergiant undergoes a type II supernova explosion, the outer layers thrown into space and what remains is the central core. The central core has become a neutron star. The neutrons in this star become degenerate due to the high density of the collapsing core. The neutron stars were predicted by Robert Oppenheimer and George Volkoff in the year 1939 on the basis of the calculated properties of a star made entirely of neutrons.

The actual structure of the star is not known completely, but there are many theoretical models that accurately describe the observations. Many of their properties are similar to those of white dwarfs. For instance, an increase in the mass of a neutron star will result in a decrease in radius with a range of radii from 10-15 km. The mass of the neutron star can be from $1.5\text{-}2.7M_{\odot}$.

The other two important properties of a neutron star is its rotation and magnetic field. A neutron star rotates hundreds or even thousands of times per second. It is due to conservation of angular momentum. When materials are thrown away from the star in arbitrary directions they pick up angular momentum. As it contracts its moment of inertia (I) decreases, so to conserve angular momentum angular velocity (ω) should increase as $I\omega$ is a constant. Another property is that, like every star has a magnetic field, neutron star also has magnetic field. The strength of the magnetic field is about 100 milliontesla.

Some neutron stars are believed to be in a binary system. These neutron stars are X-ray bursters leading to pulsars. We already discussed pulsars. The generally accepted model of a pulsar is in which the magnetic axis is tipped with respect to the axis of rotation. Very energetic particles travel along the magnetic field lines and beamed out from the magnetic poles. As the neutron star rotates around its rotation axis the beamed radiation sweeps across the earth and the pulse is detected.

Blackholes

When a star burns out of its fuel and its core mass is equal to three or more times of solar mass, it comes to the end of its life called blackhole. Blackhole is a superdense planetary material which neither emits nor reflects any light from it and appears to be black.

It is because of enormous gravitational pull and there is no opposing force inside (completely burnt out) to half this, blackhole continues to contract. A ray of light trying to leave the blackhole will be pulled back. That is even a ray of light cannot escape from it. The minimum mass of the blackhole is $3M_{\odot}$ but there is no upper limit for the mass of a blackhole. When the mass is greater than $10^6 M_\odot$, it is known as super massive blackhole.

A blackhole is described by three parameters (i) singularity, (ii) Schwarz's child radius and (iii) the event horizon.

Astrophysicists conjectured that there are three types of blackholes. They are (i) stellar blackholes (ii) primordial blackholes and (iii) super massive blackholes.

Stellar blackhole

Subramanyam Chandrasekhar established that when a star burns out of its hydrogen fuel, if the mass of the inner core is greater than $1.4M_\odot$, it undergoes gravitational collapse and becomes a neutron star with small radius. But if the mass of the inner core is greater than $3M_\odot$, the star continues its contraction due to gravitational collapse thereby overcoming the electron degeneracy and neutron degeneracy and ultimately attain a critical radius with small volume. As there is no signal can come out of this critical radius it is called event horizon.

Primordial blackholes

The study of Stephen Hawking indicates that during the time of big bang at some places matter experiences high pressure and undergoes gravitational collapse and creating micro blackholes. For instance an object of mass equal to that of earth when contracted to a radius of 1cm, then it becomes a micro blackhole. In these small blackholes quantum effect dominates. From these blackholes rays tunnel out or vapourise. This phenomenon is called hawking radiation. However there is no possibility of existence of these blackholes now. In strict sense micro blackholes are not blackholes.

Supermassive blackholes

Recent studies revealed that there are blackholes with masses 10^6-$10^9 M_\odot$ exist in our galaxy and in outer galaxies. These blackholes are called supermassive blackholes. It is shown by calculation that the age of the blackhole is proportional to the cube of its mass. Hawking shown that the age of stellar blackhole is about 10^{67} years where as that of the universe is only 10^{10} years.

There are so many theoretical models of blackholes. They are (i) Schwarzchild model (ii) Ker model (iii) Ker-Neumann model and (iv) Reissner-Nordstrom model. Schwarzchild blackholes are non-rotating and electrically neutral blackholes.

Rotating and electrically neutral blackholes are called Ker blackholes. Rotating and electrically charged blackholes are called Ker-Neumann blackholes.

In Reissner-Nordstrom model blackholes are non-rotating but electrical charged.

Among the four models Schwarzchild model is the simplest one. We discuss only about this blackhole. Before discussing blackholes in detail we have to recall what is escape velocity. Escape velocity (v_e) of an object is velocity that needs to escape the pull of gravity of a celestial object. It is given by

$$v_e = \sqrt{\frac{2GM}{R}}$$

where M is the mass of the celestial object and R is its radius.

Here our celestial object is the blackhole its escape velocity is greater than the velocity of light c. In the limiting condition we can take $v_e = c$.

$$\therefore \quad c = \sqrt{\frac{2GM}{R}}$$

$$\text{or} \quad R = \frac{2GM}{c^2}$$

Knowing G, the mass of the blackhole M and c we can calculate the radius of the blackhole. If we are dealing with a Schwarzchild blackhole, R is called Schwarzchild radius.

$$\text{i.e.} \quad R_{sc} = \frac{2GM}{c^2}$$

substituting the values of G, M and c we get

$$R_{sc} = \frac{2 \times 6.67 \times 10^{-11} \times M_{\odot}}{(3 \times 10^8)^2}$$

$$R_{sc} = \frac{2 \times 6.67 \times 10^{-11} \times 2 \times 10^{30}}{9 \times 10^{16}} = 2964\,\text{m}$$

$$R_{sc} \approx 3\text{km}.$$

This critical radius of a blackhole whose mass is equal to one solar mass is called Schwarzchild radius.

Schwarzchild blackhole

Astronomer Karl Schwarzchild applied Albert Einstein's field equation in general relativity for blackholes. **He solved field equations under the assumption that the blackhole is non-rotating and electrically neutral, thereby he could be able to describe a blackhole relativistically. This is called Schwarzchild blackhole.**

When a star contracts, due to gravitational collapse, a radius less than Schwarzchild radius becomes a blackhole we can consider Schwarzchild radius as the event horizon. What happens to the matter that is compressed into event horizon depends upon the rotation of the blackhole. If the blackhole is non-rotating (Schwarzchild black hole is non-rotating) the matter collapses to point with spherical symmetry. Technically speaking the point to which the star collapses is called singularity. At this point the volume of the matter goes to zero and gravity becomes infinity. Owing to the enormous gravity, matter is compressed to a small region of space where the identity of matter is lost and the whole laws of physics will fall flat within this region of singularity. However a rotating Ker blackhole would not have to face the phenomenon of singularity. Then the event horizon of rotating blackholes are not spherical and their poles are flattened.

Though the blackhole cannot be seen, their existence can be predicted by their influence in the neighbourhood. For instance if a blackhole is a companion of a binary star they rotate about the centre of mass. By observing changes in the visible star we can calculate the mass and the distance of the other invisible companion. If the mass is greater than $3M_{\odot}$ we can conclude that companion is a blackhole. If one of the stars in the binary is a red giant and the other is a black hole, blackhole attracts matter from the outer layers of red giant by its enormous gravity. These attracted matter rotates about the black hole forming accretion disc. The matter in the accretion disc is in the plasma state when this matter fall into event horizon, owing to their enormous speed it emits X-rays. This X-ray sources in space indicate the presence of blackholes.

Recent experiments have shown that in the constellation Cygnus there is an X-ray source, Cygnus X–1 which consists of a supergiant revolving around a small invisible companion with a mass ten times that of the sun. The companion of Cygnus X–1 is thought of to be a blackhole. Recently X-ray observatories installed in space observed so many X-ray sources in space. Among these 75% are believed to blackholes.

UNIVERSITY MODEL QUESTIONS

Section A

(*Answer questions in about* **two** *or* **three** *sentences*)

Short answer type questions

1. What is stellar evolution?
2. What is a protostar?
3. What is an embryostar?

4. What is meant by gravitational equilibrium of a star?
5. What is an evolutionary track of a star?
6. What is Hayashi phase?
7. Draw the mass-luminosity graph.
8. What is the main difference between stars of mass greater than $4M_\odot$ and less than $0.4M_\odot$ show in their evolutionary track?
9. Write down the limitations of stars masses.
10. What is a brown dwarf?
11. What are galactic clusters? Give two examples?
12. What do you meant by population I stars? Write down their pecularities.
13. What is the use of observing galactic clusters?
14. Write down three origins of triggering star formation.
15. What constitutes the internal structure of the Sun?
16. What is photosphere?
17. What is convection zone of the Sun?
18. What is radiation zone?
19. What is proton-proton chain reaction?
20. What is meant by radom walk of photons in the Sun?
21. What is diffusion time?
22. What are the informations given by very long diffusion time?
23. Brightness of Sun is very insensitive to changes in the energy production. Justify.
24. What are binary stars? Give two examples.
25. Classify the binary stars.
26. What is a spectroscopic binary? Give an example.
27. What is an eclipsing binary? Give an example.
28. What is astrometric binary? Give an example.
29. What is a visual binary?
30. Define the terms (i) separation and (ii) position angle (PA).
31. What do you understand by the term ZAMS?
32. What is the difference between zams and main sequence star?
33. Define the lifetime of main sequence star.
34. Write down the relation between lifetime of main sequence star and it mass.
35. Show graphically, how does main sequence life time vary with mass.
36. Distinguish between O-type stars and K-type stars with reference to luminosity, temperature and life time.

37. What is a red giant? Give two examples.
38. Write down the nuclear fusion reaction of helium burning in the red giant?
39. What are the ashes of helium burning?
40. What is helium flash in red giants?
41. Draw an H-R diagram indicating main sequence and the evolutionary track of red giants?
42. The main sequence on the H-R diagram is a broad band not a line. Why?
43. What are globular clusters?
44. Write down any two properties of globular clusters.
45. What is colour-magnitude H-R diagram?
46. What are the uses of colour -magnitude H-R diagram?
47. Draw a colour magnitude H-R diagram.
48. Distinguish between galactic and globular clusters.
49. What are horizontal branch stars?
50. What is the significance of turn off point in the colour-magnitude H-R diagram?
51. What is the meaning of the missing part of main sequence in the colour -magnitude H-R diagram?
52. What is a pulsar?
53. What is the main cause of pulsation in pulsars?
54. What is instability strip?
55. Show pictorially the four stages of variation of gravity and pressure during the pulsating cycle of a pulsar.
56. What was the explanation given by Arthur Eddington for a pulsating star?
57. What are Cepheid variablès?
58. What is period-luminosity relationship?
59. What is the relation between size and period of a Cepheid?
60. What is type I Cepheid?
61. What is type II Cepheid?
62. Distinguish between type I and type II Cepheids.
63. How a star dies?
64. What are the possible forms of life of a star in its old age?
65. What is asymptotic giant branch?
66. What is an AGB star?
67. What is the structure of an AGB star?
68. Show the pictorial representation of the structure of an AGB star.
69. Distinguish between a red giant and an AGB star?

70. How does an AGB star ends its life?
71. We can imagine that everything on earth is made of the stuff of stars. Justify?
72. What is a nebula?
73. What is a planetary nebula?
74. What is a white dwarf?
75. What is Chandrasekhar limit?
76. What is electron degeneracy?
77. Draw graphically the mass-radius relationship of white dwards.
78. What is white dwarf stars made of?
79. Where does white dwarf originate from?
80. What is neutron capture?
81. What is a supergiant? Give two examples?
82. Show the structure of an old high-mass star.
83. What is supernova?
84. What is meant by neutronisation?
85. What is photo-disintegration process?
86. What is a neutron star?
87. What is SNR?
88. What are the factors on which the visibility of SNR depend?
89. Classify supernovae types.
90. Distinguish between type I and II supernovae.
91. Write down three properties of neutron star.
92. How does neutron star pulsate?
93. What is a blackhole?
94. Write down the names of three parameters that describe a blackhole.
95. What are the three types of blackhole?
96. What is Schwarzchild radius?
97. What is meant by singularity?
98. What is event horizon?

Section B

(*Answer questions in a* **paragraph** *of about half a page to one page*)

Paragraph / Problem type questions

1. Briefly explain how a protostar is formed.
2. How does a star reach a hydrogen burning main sequence star?
3. What are the informations that we obtain from an H-R diagram depicted with evolutionary tracks of stars?

4. What are the four main phases of a star before it reaches the main sequence?
5. Distinguish between the radiation process and convection process based on the masses of the stars in the evolutionary track.
6. Explain the process that occurs below and above the limiting masses of star formation.
7. Explain one of the mechanisms of triggering star formation.
8. Explain how does supernova explosion triggers further star formation.
9. Explain the proton-proton chain reaction in the Sun.
10. How does energy transport from the core of the Sun to its surface?
11. Show that the diffusion time of photons of the order of 10^5 years?
12. What are binary stars? Classify and explain them?
13. Explain the processes that taking place inside the core of main sequence star.
14. Explain the formation of red giants.
15. Explain the helium burning process in red giants.
16. Explain the process of helium flash.
17. What are the informations that can be obtained from colour -magnitude H-R diagram?
18. Why do stars pulsate? Explain.
19. Explain how an AGB star is formed.
20. Explain the end of an AGB star's life?
21. How planetary nebulae are formed?
22. How does a white dwarf evolve? Explain.
23. Explain how does a supergiant form.
24. How SNR can be observed?
25. What is a stellar blackhole?
26. What is a primordial blackhole?
27. What is a supermassive blackhole?
28. How does a Schwarzchild blackhole is formed?
29. How can we detect a blackhole?

Section C

(Answer questions in about **two** *pages)*

Long answer type question (Essay)

1. Explain the birth of star?

5 GALAXIES

Introduction

Galaxy is a gravitationally bound vast system which consists of billions of stars nebulae, dusts, gases, dark matter, dark energy etc.

Galaxies are considered to be fundamental building blocks of our universe. It has been calculated that there are about 125 billion galaxies in the universe. Galaxies occupy only a small portion of the universe, remaining are spaces between galaxies. The size of a galaxy in the visible light is about 20 kpc in diameter, actually it spreads over very much larger than this. it can be seen by a radio or X-ray telescopes. Most of the visible part of the galaxy is contributed by stars. The number of stars in galaxies varies considerably. For instance in some giant galaxies there are about trillion (10^{12}) stars where as in small galaxies such as Leo I there are about a few hundred thousand. Milky way, Andromeda, Large magellanic cloud, Hubble, Ring tail, Codes, etc. are some of the galaxies.

Galaxy types

Galaxies can be broadly classified into five categories depending upon their shapes.

(i) Spiral galaxies

Spiral galaxies appear as flat white discs with yellowish bulges at their centres. The disc regions are occupied by dust and cool gas, interspersed with hotter ionised gas. Their most obvious character is the spiral arms. Example Milky way galaxy. We belong to this galaxy. It is estimated that there are 20,000 crores stars in it. Diameter of the disc of this galaxy is 10^5 ly. Our Sun is at a distance of 26,000 ly from the centre of milky way.

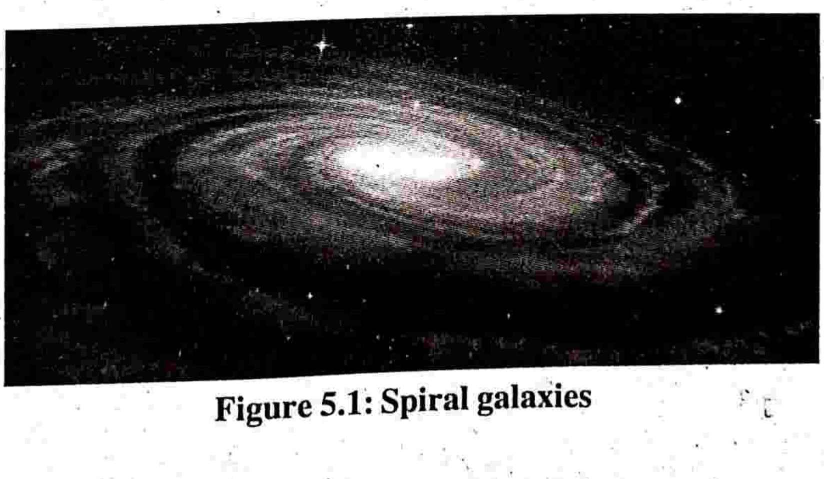


Figure 5.1: Spiral galaxies

(ii) Elliptical galaxies

Elliptical galaxies are redder and seem to be in spherical in shape with a bulge at the centre or elliptical. They contain far less cool gas and dust but very much hot ionised gas comparing to spiral galaxies. Example: Andromeda galaxy. This is the galaxy nearest to ours. It is estimated there are 40,000 corre stars in it.

Figure 5.2: Elliptical galaxies

(iii) Irregular galaxies

Galaxies that appear neither disc like nor rounded are called irregular galaxies.

Example: Large magellanic clouds.

Figure 5.3: Irregular galaxies

(iv) Barred spiral galaxies

Galaxies that exhibit a straight bar of stars that cuts across the centre with spiral arms curling away from the ends of the bars are called barred spiral galaxies.

Example: Tadpole galaxy

Figure 5.4: Barred spiral galaxies

(v) Lenticular galaxies

Galaxies that possess discs but not having spiral arms and they seem to be look like lens -shaped thus called lenticular (lens-shape) galaxies.

Example: Cart wheel galaxy.

The above classification is done by taking shapes of galaxies into account. This classification can further be subdivided by taking their brightness, the tightness of the spiral arms etc. into account.

Figure 5.5: Lenticular galaxies

Galaxy structure (spiral)

Spiral galaxies have three components. They are (i) thin disc, (ii) central bulge and (iii) the halo. Both the centre bulge and the halo together is called spheroidal component.

The thin disc of spiral galaxy extending outward from the central bulge. In the milkyway galaxy thin disc extend 50,000 ly from the centre. The disc area of all spirals contains a mixture of gas and dust called interstellar medium, but the amounts and proportions of the gas whether atomic, ionised or molecular will be different from galaxy to galaxy.

The central bulge merges smoothly into the halo, which can extend to a radius in excess of 1,00,000 ly. There is no clear boundaries for bulge and halo. Stars up to 1,00,000 ly are considered to be bulge stars and those beyond this radius are the members of the halo.

Spiral galaxies have relatively less light intensity than those of elliptical galaxies. Moreover the mass of spiral galaxies is also less than that of elliptical. Spiral galaxies have masses in the range 10^9-$10^{12} M_\odot$. It is found that 70% of spiral galaxies are barred spiral galaxies. The particular structure (bars) plays an important role in the morphology of these galaxies. The bars of the galaxy help to flow gases to the centre of the galaxy thereby setting up a situation to form new stars. As a result of this the centre bulge expands and change the shape of the galaxy.

In spiral galaxy stars are youngsters where as in elliptical stars old aged. The main reason for this is that the amount of gases is more in spherical galaxies. The amount of gases increases the possibility of formation of new stars is also increased. As gases are accumulated in spiral arms more number of new stars are formed in spiral arms. Thus spiral arms glows with much light intensity and appeared to be blue in colour. The old aged stars can also be found in spiral galaxies but they are at the middle of the bulge. The speed of revolution of old aged stars are more than that of youngster stars.

Stellar populations

We found that spiral galaxies have discs with central bulges and spiral arms. Study on aged stars in the galaxies revealed that depending upon the ages of stars disc can be divided into two layers. More gases and dust particles are accumulated at the middle layer disc. This is called thin disc. Star formation is more in this disc. The outer layer disc (bulge region) is called thick

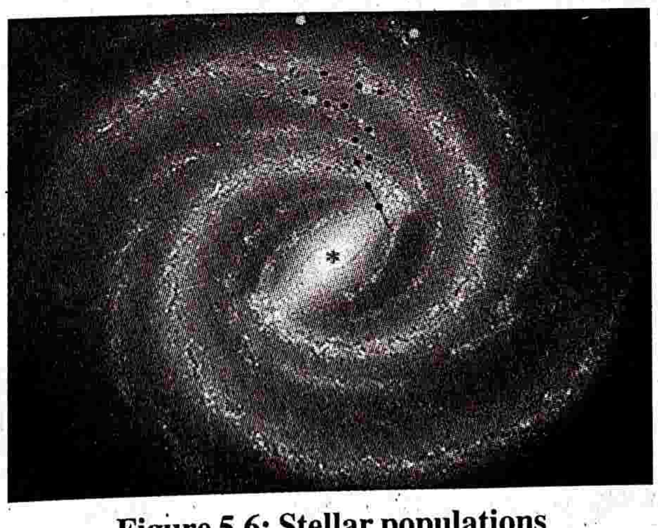


Figure 5.6: Stellar populations

disc. This disc consists of aged stars. The thin discs are at a distance of 100 pc to 325 pc from the galactic plane, where as thick discs spread over a distance of about 1500 pc.

The stars within a spiral galaxy can be classified into two by where they reside. They are population I and population II stars.

The stars in the thin discs are called population I stars. The stars in the thick disc (bulge region) are called population II stars.

Distinction between population I and II stars

1. Population I stars are young, hot and blue in colour where as population II stars are old, red giant stars and orange in colour.
2. The total mass of the population I stars about 15-30 times greater than the total mass of population II stars.

   The total mass of population II stars is about $2\times 10^9 - 4\times 10^9 M_{\odot}$.
3. 2% of the total mass of population I stars are metals (elements having atomic mass greater than that of helium's), whereas in the case of populations II stars only 0.1% of the total mass are metals.
4. The amount of light emitted by population I stars is 90 times greater than that emitted by population II stars.

Hubble classification of galaxies

The American astronomer Edwin Hubble classified galaxies on the basis of their shapes, central bulges, tightness of spiral arms and flatness called Hubble classification.

A galaxy is classified by assigning an upper case letter or letters followed either by a number or by lower case letter or letters. This classification identifies the morphology of the galaxy.

The upper case letter indicates the shape of the galaxy, the number indicates the flatness of the galaxy and the lower case letter or letters indicate the central bulge and tightness of spiral arms of the galaxy.

A spiral galaxy is represented by the letter S. If it is an ordinary spiral it is represented SA and SB represents barred spiral galaxy. It is then followed by a lower case letter a, b, c or d. If a galaxy has a large bulge and tightly wound arms it is assigned the lower case letter a. Altogether it is represented by SAa or SBa. The only difference between these two is that SAa has no bars but SBa has bars. As we go form a to d the central bulge and tightness go on decreasing. SBc is a barred spiral arms with small bulge and loosly wound spiral arms.

An elliptical galaxy is represented by the upper case letter E. This will be followed by a number 0, 1, 2.... etc. The number indicates the flatness of the galaxy. The larger the number the more flatter the galaxy. For example an EO galaxy is round and E8 galaxy is very elongated.

Galaxies neither belong to spiral nor belong to elliptical are called lenticular galaxies. A lenticular galaxy is classified as SO (both upper case). SAO is an ordinary lenticular galaxy where as SBO is a barred lenticular galaxy. In addition for galaxies intermediate between SA and SB is classified as SAB.

We have classified three types of galaxies spiral, elliptical and lenticular galaxies. Galaxies which do not fall into any of above three classification included in Pec or Irr. Pecs is the classification for peculiar galaxies which have a distorted form. Galaxies which have irregular morphology are classified as Irr. This can further be classified into two. IA and IB. IA is the irregular ordinary and IB is the barred irregular galaxy.

The Hubble classification system can be represented by a diagram as shown in figure below.

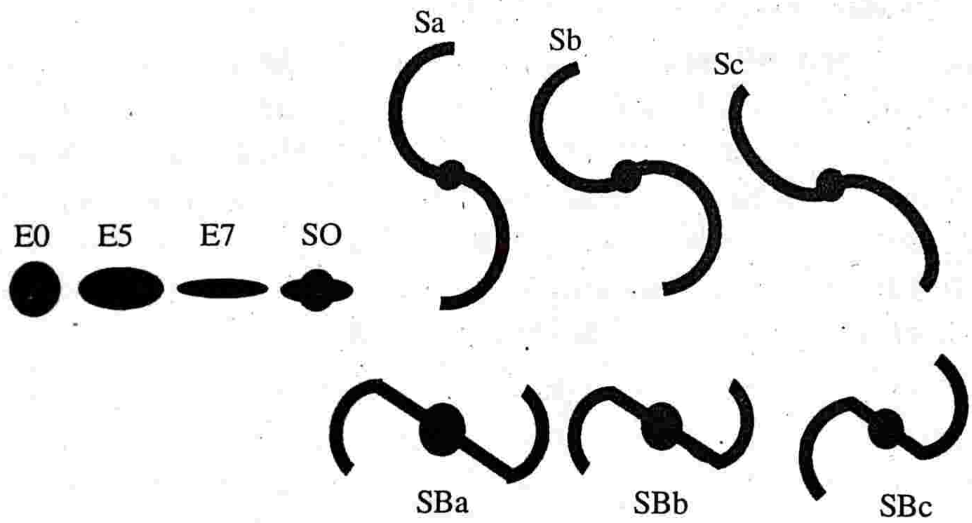


Figure 5.7: Hubble's tuning fork diagram showing the main galaxy types

Observing galaxies

All of us have seen images of galaxies in books and astronomical magazines. The images are highlighted in multicolours and seems to be highly spectacular. Before going to see galaxies, we expect dazzling images of galaxies. In reality contrary to our expectation, we can see galaxies as a pale tiny blob. We can see 9 galaxies with naked eye, binoculars or telescopes. The right location and dark sky are essential requirements to see a galaxy with naked eye. The seen galaxies will be faint and indistinct. The edge of the Milky way galaxy, M31 in Andromeda galaxy, M33 is in

Triangulum galaxy, the sides of Large magellnic cloud etc. can be seen with naked eye.

To see the real structure of galaxies we need largest telescopes and the darkest possible skies. Dark skies and a binoculars enable us to see several other galaxies. If we have a telescope the number of galaxies that can be seen will be increased considerably.

Usually galaxies with magnitude up to 8 can be seen with naked eyes. To see a galaxy of magnitude greater than 13, we require a telescope of aperture 15cm. A telescope of aperture 30cm enables us to see galaxies of magnitude 14.5. However do not expect that all parts of galaxies would be seen with these aids. In some cases only the brightest part of a galaxy will be visible, its core and spiral arms are invisible.

To trace the finer details of spiral arms of galaxies, and to locate the bulge area, faint halo etc. we need a telescope of large aperture about 2m. If our aim is just to locate a galaxy binocular or small aperture telescopes will do . If we locate a galaxy by a binocular or a naked eye there is an amazing thing behind this. That is the light that is interring our eye may have begun their journey over 100 million years ago, then there is a plethora of galaxies awaiting us.

Depending upon the brightness (magnitude) and the area of the sky the galaxy spans the visibility of the galaxy with binoculars is divided into three. They are designated as easy, moderate and difficult. **A galaxy that may be bright with magnitude 8 under normal circumstance would be visible by binoculars is designated as easy. If the galaxy of magnitude 8 covers a larger area of sky will be difficult to observe is designated as moderate. If it is not possible or very difficult to observe the galaxy is designated as difficult.**

To locate or see distant galaxies binoculars are better aids than telescopes. This is because binoculars give 3D images where as telescopes give flat images. For example to see Andromeda galaxy we require a binocular with specification 7 × 50 (7 by 50). The first number gives the magnification of the image with respect to unaided eye. The second number gives the diameter of the lens (aperture) in millimetre. There are large number of binoculars available in the market such as 7 × 50, 10 × 60, 15 ×70, 25 × 100 etc.

Astronomical catalogue

A list of objects is called as catalogue. A list of astronomical objects is called astronomical catalogue. The objects may be planetary nebulae, stars, clusters of stars, galactic clusters, globular clusters, galaxies etc. There are several types of catalogues. Here we introduce only three among them. They are Messier (French astronomer Charles Messier) catalogue, new general catalogue (NGC) and Caldwell catalogue.

The Messier catalogue consists of 110 astronomical objects where as NGC consists of 7840 astronomical objects and Caldwell contains 109 astronomical objects. For example M31. it is an astronomical object occupying 31st position in the Messier catalogue. This object occupying the 224th position of new general catalogue hence it is also named as NGC 224. This is actually a galaxy in the andromeda this is also called as Andromeda galaxy.

Note: Caldwell catalogue was compiled by Patrick Moore (Surname Caldwell) as a complement to Messier catalogue.

Finally we discuss some examples of easy, moderate and difficult galaxies belong to spiral, barred spiral and lenticular galaxies.

In the case of spiral galaxies in addition to easy, moderate and difficult designations, they exhibit a variety of views depending on their inclination to the solar system. Some will appear face on, others at a slight angle and a few completely edge on. They are symbolically represented as follows.

Face on :

Slight inclination :

Edge on :

The whirl pool galaxy is designated as easy. This galaxy is represented by

Messier 51 NGC 5194 8.4m [13.1m] SA(s) easy.

All the informations regarding the galaxy is contained in this line.

This one line representation says that the whirl pool galaxy occupies 51st position in the Messier catalogue and 5194th position in the new general catalogue. 8.4 m is the magnitude of the star and 13.1m is the surface magnitude of the galaxy. The symbol shows that is is a face on galaxy. SA(s) tells that it is an ordinary spiral galaxy. The word easy indicates that this comes under the designation easy.

This famous galaxy can be easily visible with binoculars. The galaxy appears as a small glowing patch of light with a bright star-like nucleus.

An example for moderate galaxy is

Messier 98 NGC 4192 10.1 m [13.2 m] SAB moderate.

This is a spherical barred galaxy (SAB) having magnitude 10.1 m and surface magnitude 13.2 m which is edge on. Since its designation is moderate it is difficult to locate. So a small telescope with aperture 10 cm is required.

Caldwell 26 NGC 4244 10.4 m [14 m] SA(s) difficult.

This galaxy, occupies 26th position in the Caldwell catalogue and 4244th position in the new general catalogue, having magnitude 10.4 and surface magnitude 14. It is an ordinary spiral galaxy and very difficult to locate or observe by binoculars. To observe this a telescope of aperture 20 cm is required.

Barred spiral galaxies

Here we give one example for each designation.

Caldwell 72	NGC 55	7.9 m [13.5 m]	SS	SB(s) easy
Messier 109	NGC3992	9.8 m [13.3 m]	S·S	SB(rs) moderate
Messier 91	NGC 4548	10.1 m [13.3 m]	S·S	SB(rs) difficult.

Elliptical galaxies

Messier 84 NGC 4374 9.1 m [12.3m] E1 easy.

This galaxy is located close to Virgo cluster of galaxies. It appears as a small oval patch of light when seen through the binoculars. Sometimes the bright nuclei can also be glimpsed through the binoculars.

Messier 89 NGC 4552 9.7 m [12.3 m] E0 moderate.

It is a galaxy difficult to be seen with binoculars. If it is located by binoculars it appears as a small hazy spot. But with a telescope of medium magnification this elliptical galaxy can be seen. It appears as a bright and well defined nucleus enveloped by the mistiness of the halo.

Caldwell 35 NGC 4889 11.5 m [13.4 m] E4 difficult.

This galaxy is not possible to locate by binoculars. When viewing with a telescope of aperture 20 cm we can glimpse this as tiny object. This galaxy is a dominant member of the coma galaxy cluster which contains about 1000 galaxies. This is at a distance of 350 million light years.

Lenticular galaxies

Caldwell 53 NGC 3115 8.9 m [12.6m] SOsp easy.

This is a spindle galaxy. It is easy to locate this galaxy. With small binoculars it appears as a small faint elongated cloud but with large binoculars it will display its characteristic lens shape. With telescopes of aperture 20 cm it appears as a feature less oval cloud with slight brightening towards the centre.

Caldwell 57 NGC 6822 8.8 m [14.2m] IB(s) moderate.

It is called Bernard galaxy. Even though it is a fairly bright galaxy (8.8 m), its surface brightness is low [14.2 m] it is difficult to be seen through binoculars. If it is located it just appears as indistinct glow running from east to west. This is the bar of the galaxy. However a telescopes of aperture 10 cm it can be easily located. It may be noted that it is an irregular barred (IB) galaxy.

Active galaxies and Active Galactic Nuclei (AGN)

We classified galaxies earlier primarily on the basis of their shapes. But galaxies are different not only in their shapes but also in their physical properties. This time we are going to classify galaxies on the basis of their physical properties.

In galaxies usually most of the light energy is due to stars. The wavelengths of light emitted by stars depend upon their temperature in accordance with Wien's law ($\lambda_m T$ = constant). Stars with temperature greater than 40,000K and less than 3000K are very few in galaxies. So galaxies emit wavelengths in the region from infrared to ultraviolet. But some galaxies are found to emit radio waves, ultraviolet waves, X-rays and γ-rays. At the same time they emit incredible amount of energy. These galaxies are called active galaxies.

Energy source of active galaxies

We found that active galaxies emit high energy radiations. We cannot expect these radiations from stars alone. Then, where does it come from? So astronomers conjectured that these radiations are due to some other physical phenomenon occurring in galaxies.

According to the present knowledge, we believe that it is due to the presence of supermassive blackholes in the galaxies. A galaxy hosting a blackhole emits enormous amount of energy. This is possible only due to the phenomenon of accretion. **The heavy gravity pull of supermassive blackhole attracts materials such as gas, dust and other stellar debris into it. The materials come close to the blackhole, but not fall in to blackhole, forms a flattened band of spinning matter around the event horizon called accretion disc. This phenomenon of forming accretion disc is called accretion.** The interaction between the accretion disc and the blackhole results in the production of high energy rays depending on their temperature.

Now we can redefine active galaxy as follows. A galaxy hosting a blackhole provided with accretion disc emitting enormous amount of energy and radiations such as radio waves, u.v rays, X-rays and γ-rays.

It has been estimated that an active galaxy emits about 10 million times more radio energy than a normal galaxy. The amount of light given out by an active galaxy is about 100-1000 times more than a normal galaxy.

Active Galactic Nucleus (AGN)

The central region of any active galaxy is called active galactic nucleus. It is a compact region at the centre of galaxy that has a much higher luminosity than the other parts of galaxy. The spectrum given out by AGN indicates that the luminosity is not produced by stars. Such excess non-stellar emission has been observed in the radio, infrared, optical ultraviolet, X-ray, γ-ray wave bands.

A galaxy hosting an AGN is called active galaxy. The galaxy is active in the sense that AGN holds a blackhole with accretion disc and their interaction. Though our galaxy holding a blackhole, it is not active since it has no AGN and accretion disc.

The intensity of light emitted by acive galaxies are very high. Owing to this galaxies far away from milky way can be observed through radio telescopes. Even their galactic (AGN) centres can be observed. An ordinary galaxy at this distance cannot be seen even through telescopes.

The observed characteristics of an AGN depend on several properties such as the mass of the blackhole, the rate of accumulation of accretion disc, the orientation of accretion disc, the degree of obscuration of the nucleus by dust particles.

Based on the observed characteristics of AGN, they are classified into several. Some of them are given below.

(i) Seyfert galaxies types 1 and 2

(ii) Quasars

(iii) Radio galaxies

(iv) Starbursts galaxies

Seyfert galaxies

The story of the discovery of AGN began around the year 1943. In this year astronomer Carl seyfert published a paper in which he described observations of nearby galaxies having bright nuclei that have sources of emission lines. Galaxies observed as part of this study included NGC 1068, NGC 4151, NGC 3516 and NGC 7469. Active galaxies such as these are known as Seyfert galaxies in honour of Seyfert's pioneering work.

A galaxy which contains AGN is also called as host galaxy. Depending upon the shape host galaxy it is divided into two. They are seyfert galaxy and radio galaxy. Seyfert galaxies are spiral where as radio galaxies are elliptical.

Depending upon the nature of emission lines in the spectrum of AGN they are divided into seyfert 1 and seyfert 2 galaxies. If the spectrum contains 8 broad and narro emission lines they are called seyfert 1 galaxies.

If the spectrum contains only narrow emission lines they are called syfert 2 galaxies.

Quasars

The word quasar is the abbreviated form of quasi stellar radio sources. The name implies start like emitters of radiowaves. This name was given in the year 1960. The first quasar 3c273 was discovered by Maarten Schmidt at Halo observatories.

Quasar is a super luminous AGN at billions of light years away from us. All quasars emit X-rays in abundance. Only 5% quasars emit radio waves. Since quasars are far away from us they seen to be like stars when looking through telescopes. A quasar has mainly three parts 1) The central region (2) Jets and (3) Lobes. Jets emanating from the central region onto two sides and end up in lobes. The lobes are made of electrons moving with velocities about the velocity of light. When these electrons moving through the magnetic field they get accelerated and producing radio waves. This phenomenon is called as synchrotron radiation. It may be noted that quasars exhibit red shifts.

Definition of quasar

A quasar is an active galaxy whose AGN is extremely luminous in which there is a supermassive blackhole with mass ranging from millions to billions times the mass of the Sun surrounded by an accretion disc.

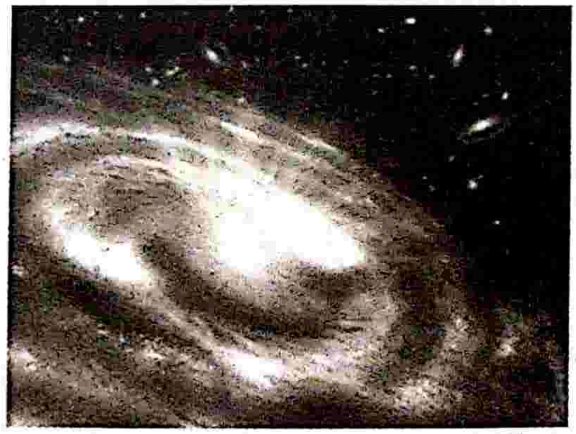


Figure 5.8: Quasar

Discovery of quasars changed our view of the universe.

1. Opened up study of the distant universe
2. Led to realise that super massive blackholes exist
3. Gravitational energy is the source for quasars and AGN

Radio galaxies

All quasars do not emit radio waves. Such quasars are called radio quite quasars. **Galaxies which emit radio waves are called radio galaxies.** This can be identified from the spectrum. Radio galaxies are spiral galaxies. Cygnus A and Centarus A are radio galaxies.

Star bursts galaxies

Star burst galaxies are galaxies formed when compared to others. For instance our Milky way galaxy converts gases of mass nearly $3M_{\odot}$ into stars in every year, whereas star bursts galaxies convert gases into stars about 100 times greater than by milky way. As a result the dust particles are much hotter and they emit infrared rays. Thus they are called ultra luminous infrared galaxy (ULIRG). Most of the star bursts galaxies are discovered by infrared astronomical satellites of NASA.

Gravitational lensing

In between a distant galaxy (light source) and an observer on earth massive objects (such as stars, blackholes and other galaxies) are distributed. **As light from the galaxy passes by these massive objects, the gravitational pull can bend the light rays. Here the matter between the galaxy and the observer acts like lens. This phenomenon is called gravitational lensing.**

According to Einsteins general theory of relativity, gravity can effect light rays. Thus light rays coming from a distant galaxy or a quasar when passes close to another galaxy or a blackhole, light rays get deviated from their path. This leads to different interesting effects.

1. When we observe a distant galaxy source we get two images of the source one is the direct image the other one is image produced by gravitational lens.

   This possibility was firstly predicted by Fritz Zwicky in the year 1937. In 1979 Dennis Walsh, Bob Carswell and Ray Weyman discovered a binary quasar named 0957 + 561. When they analysed their spectra and red shift they were found to be the same. The only difference is in their image positions. They were actually proving the phenomenon of gravitational lensing indirectly. That is 0957 + 561 is not a binary but a single one.

2. If the object behaving like a lens is very big and massive, for example galaxy cluster, its produces several images of the distant galaxy on a ring. This is called Einstein ring. Several such rings were observed when a distant blue galaxy was photographed by Hubble telescope.

3. If the mass of the object (lens) lying in the path of the light rays is small and heavy (blackhole), the lens focuses faint rays making it brighter. This is called micro lensing. Since the lensing process increases the brightness of light rays gravitational lens is also called as gravitational telescope.

All gravitational lenses discovered so far are incidentally. Gravitational lensing is used to probe the distribution of matter in the galaxies and clusters of galaxies. This enables us to observe the distant universe.

Now a days astronomers are searching for gravitational lenses to gather more informations about the cosmos.

Hubble's law

Hubble's law is one of the most important concepts in astrophysics. During the early part of the 20th century Edwin Hubble began to take the spectra of distant galaxies. Analysing the spectra he could understand that all galaxies exhibit red shift. Red shift is the phenomenon of shifting the spectral lines towards the red end of the spectrum. This indicates that galaxies are moving away from us. By observing large number of galaxies he arrived at Hubble's law. It states that the recessional velocity (v) of a galaxy is directly proportional to its distance (d)

i.e. $$v \propto d$$

or $$v = H_0 d$$

Where H_0 is called Hubble constant. the value of H_0 is found to be $70 kms^{-1}(Mpc)^{-1}$ (present value)

The red shift and the recessional velocity can be determined experimentally easily. this enabled us to know more about the cosmos and big-bang.

Note: The red shift (z) is given by

$$z = \frac{\lambda_{obsrved} - \lambda_{rest}}{\lambda_{rest}}$$

The red shift and Hubble's law we already dealt with in detail.

Clusters of galaxies

Most of the galaxies are found in clusters and single galaxies are very rare. If the cluster contains a few galaxies called small clusters and if it contains thousands of galaxies they are called giant clusters.

Small clusters occupy a region of space spread over only 1Mpc where as giant clusters occupy a region of space spread over about 10 Mpc.

The Milky way galaxy is a member of a small cluster called the local group containing more than three dozen other galaxies. There is another type of classification.

Clusters can be divided into two types. They are rich clusters and poor clusters. The rich cluster consists of more than 1000 galaxies and cover an area about 3Mpc in diameter. In this cluster galaxies are concentrated at the centre of cluster. At the centre there may one or two giant elliptical galaxies. For example Virgo cluster is a rich cluster with the giant elliptical M87 at its centre.

Poor clusters contain fewer than a 1000 members and as big as a rich cluster.

It is observed that rich clusters contain about 80-90% E type and 10% S0 type galaxies. Whereas poor clusters contain large proportion of spiral galaxies. The galaxies in isolation (those not in clusters) 80-90% are spiral galaxies. Observations of elliptical galaxies indicate that they are formed due to merging of spiral galaxies.

The evolution of galaxies is still not fully understood yet. However in clusters of galaxies collisions, merges and close encounters can obviously cause bursts of star formation and dramatic tidal disruption. It is due to this changes in clusters astronomers now believe that our Milky way is gradually swallowing Large magallenic cloud galaxy which is next to Andromeda galaxy.

UNIVERSITY MODEL QUESTIONS

Section A

(*Answer questions in about* **two** *or* **three** *sentences*)

Short answer type questions

1. What is a galaxy?
2. Give the names of any four galaxies.
3. Classify galaxies on the basis of their shape.
4. What is a spiral galaxy?
5. What is an elliptical galaxy?
6. What are irregular galaxies?
7. What are barred spiral galaxies?
8. What are lenticular galaxies?
9. Distinguish between spiral and elliptical galaxies.
10. The age of the stars in spiral galaxies are relatively younger. Justify.
11. What is meant by population I stars?
12. What is meant by population II stars?
13. What are the two essential conditions to observe galaxies with naked eye?
14. What are the designations given to galaxies that can be observed by a binocular?
15. When the galaxy is designated as easy?
16. When the galaxy is designated as moderate?
17. When the galaxy is designated as difficult?
18. What does the representation "Messier 81 NGC 3031 6.9m (13m) S·S. SA Easy" Mean?
19. A galaxy which is difficult to observe by binoculars occupying 48th position in the Caldwell catalogue and 2775 position in new general catalogue having magnitude 10 and surface magnitude 13.1, which is face on and ordinary spiral. Represent this galaxy?

20. What are active galaxies?
21. What is an active galactic nucleus ?
22. What is an accretion disc?
23. Define the phenomenon of accretion.
24. What all factors on which characteristics of an AGN depend?
25. Give the names of three active galaxies.
26. What is a seyfert galaxy?
27. What is a quasar?
28. Which are the three main parts of a quasar?
29. What are radio galaxies?
30. What are star bursts galaxies?
31. What is a gravitational lens?
32. What is gravitational lensing?
33. What is meant by red shift?
34. What is Hubble's law?
35. What is meant by clusters of galaxies?

Section B

(*Answer questions in a* **paragraph** *of about half a page to one page*)

Paragraph / Problem type questions

1. What is the structure of spiral galaxy?
2. Distinguish between population I and II stars.
3. What is Hubble classification of galaxies?
4. Distinguish between easy, moderate and difficult designation of galaxies.
5. Give a brief account of three astronomical catalogues.
6. How does an active galaxy get extra energy?
7. What are the effects of gravitational lensing?
8. Distinguish between a rich cluster and a poor cluster.
9. How does red shift and Hubble's law are related?
10. How did quasars discovery change our view of the universe?

Section C

(*Answer questions in about* **One** to **two** *pages*)

Long answer type questions (Essays)

1. Discuss galaxies in detail.
   (Definition, types and structure)
2. Discuss gravitational lensing in detail.
   (Definition, their effects and binary quasar)